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ON A NEW CLUSTER OF NEBULAE IN PISCES

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A. *Introduction.*—Clusters of nebulae represent the largest aggregations of matter known. It may be expected that an investigation of their distribution in space, of their composition and physical properties will throw new light on such problems as the determination of nebular masses,[1] on the red shift of light from nebulae and on the evolution of stars and nebulae as well as the evolution of the universe as a whole. For these reasons it was thought desirable to undertake a more concentrated study of clusters of nebulae than hitherto attempted. The first step of this program consists in securing more complete data on the number and the distribution of groups of nebulae containing two, three or more members up to clusters of many hundred nebulae. The new 18-inch Schmidt telescope of the California Institute of Technology promises to be an ideal instrument for the investigation of groupings of nebulae brighter than the sixteenth magnitude.

B. *The 18-inch Schmidt Telescope.*—This instrument was built in the shops of the Institute under the direction of Dr. John A. Anderson. Some of its characteristic features are: effective aperture 18 inches, focal ratio $F/2$, diameter of field actually used is about 9.5 degrees; this field is free of coma and astigmatism to its very edge. The limiting magnitude for stellar images lies close to $m = 17.5$ and is reached with exposures of about forty minutes.

The first photographs with the new telescopes were taken on Sept. 5, 1936. During the following two months I obtained over one hundred photographs with exposures of ten to thirty minutes covering about one-fifth of the entire sky. Since then the weather has been mostly bad and it will not be possible to check the results obtained in the first fast survey until the fall of 1937. The results described in the following should therefore be regarded as preliminary. These results are presented here for two reasons. In the first place they illustrate the kind of work which may be

done with a Schmidt camera of the type described. Secondly they outline a part of the proposed program on clusters of nebulae.

C. *Distribution of Nebulae in the New Cluster.*—Several new concentrations of nebulae were spotted in the first survey with the Schmidt camera. A large swarm of nebulae in Pisces promises to be of particular interest. In contradistinction to most of the clusters previously described[2] the extension of the new cluster is very great and its boundary surface is characterized by extreme diameters whose ratio is about of the order 1:5. In figure 1 the dots indicate nebulae which can easily be spotted on ten minute exposures. The faintest among them have an apparent magnitude of about $m = 16$. The greatest concentrations of nebulae occur in the groups C_1, C_2, C_3 and C_4. No nebulae are plotted in the region R because no good picture was available. The nebulae brighter than $m = 13$ are marked with their N. G. C. numbers 598, 404, 670 and 672.

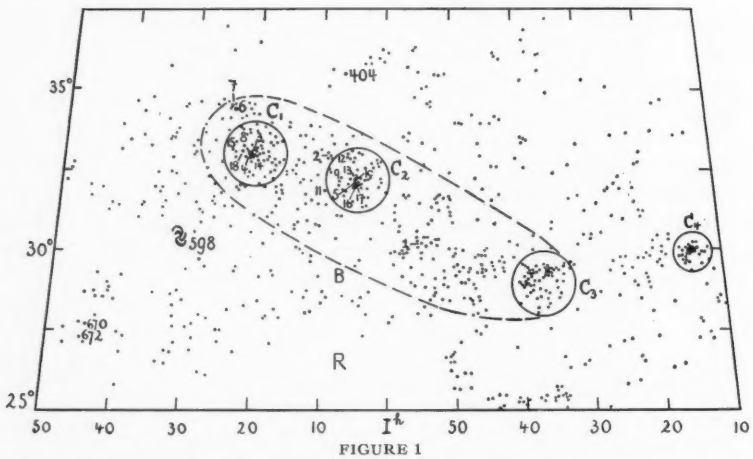

FIGURE 1

Positions in the figure are for the year 1950. No effort, however, has been made to plot the position of every nebula accurately.

According to Hubble's counts the average number N_m of nebulae per square degree near the galactic pole whose brightness is greater than m is given by [3]

$$\log_{10} N = 0.6\, m - 9.1. \qquad (1)$$

However, the actual number of nebulae observed is smaller because of the interference of obscuring matter. At the galactic latitude β a correction for partial obscuration must be added, so that the number of nebulae observed is N_β.

$$\log N_\beta = \log N + 0.15\,(1 - \operatorname{cosec} \beta). \qquad (2)$$

Since for the new cluster approximately $\beta = -30°$, we obtain for the limiting magnitude $m = 16$ the number $N_\beta = 2.2$ nebulae per square degree in this region. This is just about the number of nebulae which can readily be recognized on my ten minute exposures in the normal field. Inside the tentative boundary line B of the cluster (Fig. 1) the count rises to as high as $N_\beta = 30$.

The total number of nebulae brighter than $m = 16$ which belong to the cluster is of the order $N = 120$.

D. *Approximate Magnitudes of the Brightest Nebulae in the Cluster.*— Unfortunately extra focal images of the present Schmidt telescope are not very suitable for the comparison of nebular magnitudes with magnitudes of standard stars. Because of zones in the optical system, the extrafocal images are not sufficiently uniform. I found, however, that by appropriately timed guiding imitating the motion of a Schraffier-Kassette, the extrafocal images may be smeared over square areas in such a manner that over a wide field these squares show uniform blackening. Since the calibrated stars of the Mt. Wilson selected area No. 45 can be photographed on the same film with a major part of the neighboring Pisces cluster I was able to secure fair preliminary magnitudes for the eighteen brightest nebulae of this cluster. I am greatly indebted to Dr. F. H. Seares of the Mt. Wilson Observatory for the use of his tables of standard stars and his magnificent marked photographs of selected areas which enabled me to tie speedily nebular magnitudes to the magnitudes of standard stars. The photographic magnitudes of the eighteen brightest nebulae, marked 1, 2,... 18 in figure 1 are as follows.

TABLE 1

NO.	N. G. C.	m_p	NO.	N. G. C.	m_p
1	315	13.2	10	380	13.95
2	410	13.3	11	420	14.05
3	499	13.5	12	392	14.05
4	507	13.55	13	379	14.10
5	383	13.55	14	494	14.20
6	529	13.60	15	528	14.20
7	536	13.60	16	385	14.20
8	513	13.70	17	384	14.20
9	403	13.80	18	517	14.25

The N. G. C. numbers of the marked nebulae are given in the second column. The greatest concentrations of nebulae occur inside of the circles marked C_1, C_2, C_3. Whether or not all of these groups, as well as perhaps an additional group C_4, really form one giant system of nebulae can only be decided by further investigations.

From Hubble's counts of more distant nebulae it follows that the obscuration all over the new cluster is approximately uniform and amounts

to a loss of about 0.25 magnitudes relative to the galactic pole. If we reduce our magnitudes to the pole we must therefore add $\Delta m = -0.25$ to the value of m given in table 1.

E. *Distance of the New Cluster.*—According to Hubble[3] the fifth brightest nebula of a cluster on the average has an absolute magnitude $M_5 = -16.4$. In our list the fifth brightest nebula is of the apparent magnitude $m_5 = 13.6$. In order to determine the distance we must correct this magnitude for total obscuration. The reduction to the galactic pole requires $\Delta m = -0.25$. In addition 0.25 magnitudes are lost because of obscuration at the pole, so that $m_5 = 13.6 - 0.5 = 13.1$. The distance p in parsecs of the new cluster therefore follows from

$$\log_{10} p = 1 + (m - M)/5 = 6.9. \tag{3}$$

$$p = 7.94 \times 10^6 \text{ parsecs} = 25.8 \times 10^6 \text{ light-years}. \tag{4}$$

Since the long axis of the cluster extends over more than ten degrees, the actual extension must be greater than four million light-years whereas the shortest axis is of the order of one million light-years.

It should further be noted that the western end of the cluster (Andromeda end) seems to lie at a greater distance than the eastern Pisces end since there is a decrease in brightness of the cluster nebulae going from east to west. The long axis of the cluster therefore may form with the line of sight an angle considerably different from a right angle.

F. *Apparent Velocities of Recession of Nebulae in the New Cluster.*—Hubble[3] gives the following relation for the apparent velocity of recession v of a cluster in terms of the magnitude m_5 of its fifth brightest nebula.

$$\log v = 0.2\, m_5 + 1.025. \tag{5}$$

In our case, with $m_5 = 13.1$, we should expect

$$v = 4400 \text{ km./sec.} \tag{6}$$

This value refers to the presumably nearer eastern end of the cluster.

Mr. M. L. Humason of the Mt. Wilson Observatory has kindly consented to measure the red shifts of some of the more prominent nebulae in the new cluster. But so far weather conditions have been most unfavorable. Some data, however, are already available from previous measurements which roughly check the conclusions drawn in this paper.

Hubble and Humason had already previously recognized some groupings of nebulae in Pisces and Andromeda.[2,4] Most conspicious among these groups is what they designate as the *Pisces group*, with the coördinates R. A. $1^h\ 3.5^m$, Decl. $+\ 32°$, 1930. In figure 1 the circle C_2 encloses this group. Hubble and Humason[2] are of the opinion that "this is not a cluster, but a group of some 25 elliptical nebulae which stands out from the ap-

proximately uniform background of field nebulae." Since the 100-inch plates cover less than one square degree it is readily understandable that these plates do not directly reveal the fact that the Pisces group may be part of a large cluster covering perhaps more than 15 square degrees. The value of powerful wide angle telescopes such as the new Schmidt camera for investigations of nearby clusters of nebulae becomes thus apparent.

Hubble and Humason[2] give as the average v of four nebulae in the Pisces group

$$v = 4630 \text{ km./sec.} \qquad (7)$$

with a range of 500 km./sec. between the individual values. This constitutes a fair check of the previous result (6).

Humason[4] gives additional apparent velocities for nebulae in the group enclosed by the circle C_4.

	R. A. (1950)	Decl.	v
Group C_4	$0^h\ 15.9^m$	29°50′	6560 km./sec.

Whether or not this group has any mechanical relation to the Pisces cluster must be determined by further observations.

G. *Concluding Remarks.*—Assuming that the range in brightness in clusters of nebulae is about five magnitudes, the total number of nebulae in the Pisces cluster may be estimated as between 300 and 400. As already mentioned the most interesting feature of this new cluster is its large extension and its asymmetrical flat shape which suggests that the cluster as a whole is rotating. It will be of interest to test this point. Among the already known clusters the new cluster has perhaps its closest analogue in the clustering in Centaurus described by H. Shapley.[6] The analysis of these two clusters also suggests that the well-known Virgo cluster may not be confined to the relatively small region ascribed to it ordinarily. The concentration of nebulae in Virgo plus the northern and southern extensions of this concentration into Canes Venatici, Ursa Major and Hydra, respectively, may form an extended cluster of the type described in this paper.

On finishing this study I ran into a preliminary notice by C. W. Tombaugh and C. O. Lampland[7] who announce the discovery, with the 13-inch Lawrence Lowell telescope of Flagstaff, of a considerable number of clusters of nebulae ranging from 20 to 90 members. These findings confirm the impression gained in my first fast survey with the Schmidt camera, namely, that our knowledge of the clustering of nebulae will be greatly enriched by the use of powerful wide angle telescopes.

My sincere thanks are due to the observatory council of the California Institute of Technology for the use of the new Schmidt camera as well as for the provision of all the necessary working and living facilities at the new observatory site on Mt. Palomar. I am also greatly indebted to

Dr. W. Baade who has given much of his time to teach me some elements of practical astronomy.

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*ELECTROKINETICS. XIX. INTERFACIAL ENERGY AND THE MOLECULAR STRUCTURE OF ORGANIC COMPOUNDS. V. THE ELECTRIC MOMENT OF Al_2O_3: BENZENE-NITROBENZENE INTERFACE**

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Martin and Gortner[1] and Jensen and Gortner[2] have shown that an intimate relationship exists between the structure of organic molecules in a liquid and the electrokinetic potential which exists at the interface between a solid and that liquid. The greater the polarity or the dipole moment of the liquid, the greater is the potential which develops. Since benzene possesses no permanent dipole moment, no potential develops at the solid-liquid interface. On the other hand, benzene derivatives in which some radical has been substituted for a hydrogen all show orientation and give rise to an electrokinetic potential. The potential developed by nitrobenzene is of large magnitude.[1] Accordingly it seemed of interest to study the behavior of a mixed liquid system containing both polar and non-polar molecules.

The streaming potential technique was used and the sample of Al_2O_3 for the diaphragm was from the same lot employed by Jensen and Gortner in their earlier studies. The large resistances encountered were measured using a quadrant electrometer as a null instrument. We were unable to find a complete series of concentration: dielectric constant values for the system benzene–nitrobenzene. Accordingly the data are recorded only as $\frac{H\kappa\eta}{P}$ where H = e. m. f. developed by streaming a liquid of viscosity η through the diaphragm under hydrostatic pressure P and κ is the specific conductivity of the liquid in the pores of the diaphragm.

This value when converted into electrostatic units is the electric moment (μ) per unit area of the double layer.[3] The electrokinetic potential can be calculated from this value when the dielectric constant (ϵ) is known.

$$\zeta = \frac{4\pi H\kappa\eta}{P\epsilon}.$$

The data for the measurements for benzene, nitrobenzene and mixtures of these liquids at the Al_2O_3 interface are shown in table 1 and figure 1. The table and graph also include three measurements on benzene phenol mixtures.

Figure 1 shows that up to about 0.2 mol fraction the nitrobenzene contributes only slightly to the development of a double layer, whereas a similar increment in excess of 0.5 mol fraction produces an enormous effect upon the electrical phenomenon at the interface.

The question arises as to how such divergent effects are to be interpreted. One interpretation might be that a completely oriented layer is not formed until the surface becomes close packed and that saturation of the surface with nitrobenzene molecules is not reached except in practically pure nitrobenzene. An alternate explanation suggests the formation of a bimolecular compound between the benzene and nitrobenzene whereby the polar group of the nitrobenzene is prevented from exhibiting its electrical forces under the technique employed. Some evidence for the existence of such a bimolecular compound is given by the fact[4] that a eutectic mixture of benzene-nitrobenzene occurs at 51.3 mol per cent of nitrobenzene. If this explanation be the cause of our results, then electrokinetic measurements may serve as an additional tool for the detection of molecular species

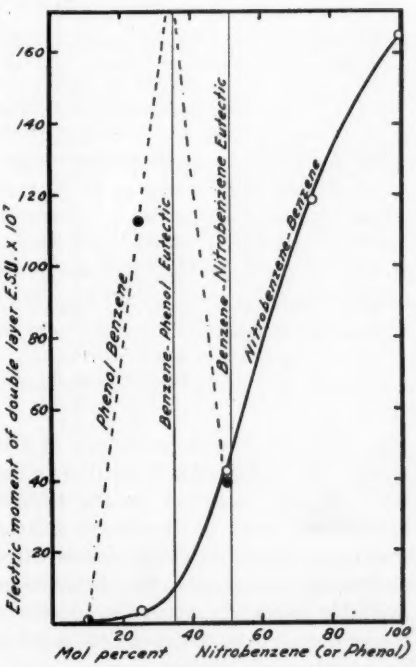

FIGURE 1

Showing the electric moment of the double layer at the interface between Al_2O_3 and benzene, nitrobenzene and various mixtures of these substances. (Certain data on phenol-benzene mixtures are also indicated.)

which it is impossible to demonstrate by direct isolation. Support for this suggestion is found in the benzene-phenol data. In this system a eutectic occurs[4] at 34.8 mol per cent phenol. This eutectic is apparently a compound of two mols of phenol with one of benzene, corresponding to a eutectic at 21.06 mol per cent of the diphenol molecule. The data in table 1

TABLE 1

ELECTROKINETIC DATA AND THE CALCULATED ELECTRIC MOMENT OF THE DOUBLE LAYER AT THE INTERFACE BETWEEN Al_2O_3 AND PURE BENZENE, PURE NITROBENZENE AND CERTAIN MIXTURES OF THESE LIQUIDS

(Including similar data on three mixtures of phenol-benzene)

VOLUME % NITRO-BENZENE	MOL % NITRO-BENZENE	H/P MV./CM. Hg	SPECIFIC CONDUCTIVITY MHOS. $\times 10^9$	VISCOSITY POISE	$\frac{H\kappa\eta}{P} \times 10^{10}$	ELECTRIC MOMENT (μ) E. S. U. $\times 10^{17}$
0	0.00	0.0	....	0.00648	0.0	0.000
10	8.82	1.59	1.13	0.00680	1.23	0.274
25	22.50	15.90	1.23	0.00764	14.95	3.35
50	46.60	141.0	1.42	0.00938	188.0	42.2
75	72.40	252.7	1.76	0.01196	532.0	119.0
100	100.00	212.0	2.11	0.01652	737.0	165.0
mol per cent phenol						
	10	38.0	0.436	0.00750	12.4	2.78
	25	20.0	25.60	0.00967	503.0	113.0
	50	9.75	12.10	0.01481	176.0	39.4

were secured before the possibility of a molecular compound as influencing results was considered and accordingly the data are incomplete. The dotted line in figure 1 is drawn on the hypothesis that the maximum electric moment will occur at the eutectic mixture. This would be the case if the compound $2C_6H_5OH:C_6H_6$ is more strongly adsorbed and more surface electrically active than is the phenol molecule itself. In any event the data available indicate a maximum electric moment for phenol-benzene at a concentration close to the eutectic mixture. This is the reverse behavior of the nitrobenzene–benzene system.

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APPLICATION OF A NEW MATHEMATICAL METHOD TO VIBRATION-ROTATION INTERACTION

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The question of vibration-rotation coupling has been considered by various authors using different kinds of potential functions. Only in the simple case of the Kratzer potential has the problem been treated exactly.[1,2,3] In all others the procedure has been to expand the rotational term in a power series in the neighborhood of the equilibrium point, keeping only the first few powers so that the usual perturbation calculation or the W. K. B. method[4] could be applied. The objection to this is that the expansions are used in regions where they are not valid. In the perturbation method, for example, the terms of the expansion which are kept are integrated over all space even though the expansion diverges outside a small circle. For these reasons we present a different approach which makes it possible to deal with the exact form of the problem.

The equation for the radial component F of the eigenfunction of a rotating oscillator is:

$$\frac{d^2F}{d\rho^2} + \left[\frac{E}{B_e} - \frac{V}{B_e} - \frac{u}{\rho^2}\right]F = 0. \tag{1}$$

E is the term value in cm.$^{-1}$, $\rho = r/r_e$, V is the potential function expressed in suitable units, $u = J(J+1)$ and the other symbols have their standard significance. The function F is so defined that for two states v' and v'' the $(v'v'')$ matrix component of any quantity a is given by:

$$a_{v'v''} = \int_0^\infty F^*_{v'} a F_{v''} d\rho.$$

We shall consider the case of the harmonic oscillator, for which $V = \frac{1}{4}(\omega^2/B_e)(\rho-1)^2$. If we let $\lambda = E/\omega_e - {}^1/_2$, $\alpha = B_e/\omega_e$ and $F = z\exp[-(\rho-1)^2/4\alpha]$, the latter substitution indicated by the asymptotic behavior of the function F, we obtain:

$$\frac{d^2z}{d\rho^2} - \frac{1}{\alpha}(\rho-1)\frac{dz}{d\rho} + \left[\frac{\lambda}{\alpha} - \frac{u}{\rho^2}\right]z = 0. \tag{2}$$

It would seem at first that this could be treated by standard methods as shown by Sommerfeld,[3] a solution of equation (2) by means of infinite

series leads to a three-term recurrence formula from which one may usually determine characteristic values. Unfortunately, however, the procedure developed by A. H. Wilson[5] and others for dealing with the three-term recurrence formula is inapplicable in the present case, and our efforts to modify the Wilson treatment to make it suitable for this problem have been unsuccessful.

Since $\alpha < .01$ for all known molecules, we can expand the solution in terms of α as a small parameter. With the new variable $\xi = \rho - 1$ ordinarily used in vibration problems, equation (2) becomes:

$$\frac{d^2z}{d\xi^2} - \frac{\xi}{\alpha}\frac{dz}{d\xi} + \left[\frac{\lambda}{\alpha} - \frac{u}{(1+\xi)^2}\right]z = 0. \tag{3}$$

When $J = 0$ the characteristic value is an integer v and the unperturbed solution[6] (already a function of α !) is:[7]

$$y = \sum_{k=0} \frac{(-\alpha)^k v\,!\,\xi^{v-2k}}{2^k k\,!\,(v-2k)!}. \tag{4}$$

We introduce the substitution[8]

$$z = y \exp[-\alpha u \int \varphi d\xi],$$

which gives

$$\left[\alpha\frac{d\varphi}{d\xi} - \xi\varphi - \alpha^2 u\varphi^2 - \frac{\lambda - v}{\alpha u} + \frac{1}{(1+\xi)^2}\right]y + 2\alpha\varphi\frac{dy}{d\xi} = 0. \tag{5}$$

If we assume now that

$$\lambda = v + \alpha u \sum_{k=0}^{\infty} \alpha^k \lambda_k \text{ and } \varphi = \sum_{k=0}^{\infty} \alpha^k \varphi_k$$

and substitute for y the explicit form (4), we may set the coefficients of successive powers of α in equation (5) equal to zero and thus obtain a set of recurrence formulas for the φ_k's:

$$\sum_{m=0}^{k-1} \frac{(-1)^m v!\,\xi^{v-2m}}{(v-2m)\,!\,2^m m!}\left\{\frac{d\varphi_{k-m-1}}{d\xi} - \xi\varphi_{k-m} - u\sum_{n=0}^{k-m-2}\varphi_{k-m-n-2}\varphi_n - \lambda_{k-m}\right\}$$

$$+ 2\sum_{m=0}^{k-1} \frac{(-1)^m v\,!\,\xi^{v-2m-1}}{(v-2m-1)\,!\,2^m m!}\varphi_{k-m-1} = 0, \tag{6}$$

if $k > 0$ and:

$$-\xi\varphi_0 - \lambda_0 + \frac{1}{(1+\xi)^2} = 0, \tag{7}$$

if $k = 0$. The eigenfunction F must obviously be finite in the interval $-1 \leqslant \xi < \infty$ and vanish at the limits.[6] In terms of φ this is equivalent to postulating that

$$\int \varphi d\xi \longrightarrow \infty \quad \text{as } \xi \longrightarrow -1.$$

The integral may not tend to $-\infty$ since that would make z infinite; this requirement applies not only to the sum φ but also to each φ_k. However, as $\xi \longrightarrow \infty$ we may allow $\int \varphi d\xi \longrightarrow -\infty$ if the approach is sufficiently slow, so that F still tends to zero because of the factor $\exp[-\xi^2/4\alpha]$.

For the actual determination of energy values we start with the state $v = 0$ for which $y = 1$ and equation (6) reduces to:

$$\frac{d\varphi_{k-1}}{d\xi} - \xi\varphi_k - u\sum_{n=0}^{k-2} \varphi_{k-n-2}\,\varphi_n - \lambda_k = 0. \tag{6a}$$

Equation (7) remains unchanged. It may be rewritten as:[9]

$$\varphi_0 = \frac{1-\lambda_0}{\xi} - \frac{1}{1+\xi} - \frac{1}{(1+\xi)^2}$$

which gives:

$$\int \varphi_0 d\xi = (1-\lambda_0)\log\xi - \log(1+\xi) + \frac{1}{1+\xi}. \tag{8a}$$

The conditions for the finiteness of F are satisfied automatically at the boundaries. However, at the singular point[10] $\xi = 0$ the integral in equation (8a) will tend to $-\infty$ unless:

$$1 - \lambda_0 \leqslant 0. \tag{9a}$$

We pass now to the next approximation

$$\varphi_1 = \frac{\lambda_0 - 1}{\xi^3} + \frac{3-\lambda_1}{\xi} - \frac{3}{1+\xi} - \frac{3}{(1+\xi)^2} - \frac{2}{(1+\xi)^3}. \tag{10a}$$

Again a difficulty arises only at the point $\xi = 0$, where $\int \varphi_1 d\xi \longrightarrow -\infty$ unless:

$$1 - \lambda_0 \geqslant 0. \tag{11a}$$

Combined with (9a) this determines the characteristic value in this approximation as:

$$\lambda_0 = 1. \tag{12a}$$

Since the term in $1/\xi^3$ in φ_1 vanishes, the behavior of this function at $\xi = 0$ is determined by the coefficient of $1/\xi$. In strict analogy with the reasoning given for λ_0 we find from φ_1 that $3 - \lambda_1 \leqslant 0$ and from φ_2 that $3 - \lambda_1 \geqslant 0$, hence $\lambda_1 = 3$. The form of equation (6a) indicates that if there are no terms in $1/\xi$ in φ_{k-1}, there will be no powers of $1/\xi$ higher than 1 in φ_k, etc. We have made calculations up to the sixth order and obtained the following results for the characteristic value and the characteristic function of the state $v = 0$:

$$\lambda = \alpha u\{1 + 3\alpha u + \alpha^2(15 - 4u) + \alpha^3(105 - 57u) + \alpha^4(945 - 716u + 48u^2)\} \tag{13a}$$

$$z = y(1+\xi)^{\lambda}\exp\left\{-\frac{\lambda}{1+\xi} - \frac{\alpha^2 u}{(1+\xi)^2}\left[1 + 3\alpha(2 - \tfrac{1}{2}u) + \alpha^2(45 - 47u/2) + \alpha^3(420 - 217u/2 + 20u^2)\right] - \frac{\alpha^3 u}{(1+\xi)^3}[2 - u/3 + \alpha(20 - 9u) + \alpha^2(210 - 428u/3 + 8u^2)] - \frac{\alpha^4 u}{(1+\xi)^4}[6 - 2u + \alpha(90 - 53u + 5u^2/2)] - \frac{2\alpha^5 u}{(1+\xi)^5}(12 - 28u/5 + u^2/5)\right\}. \tag{14a}$$

It may be pointed out that the value of λ agrees with that obtained by Dunham.[4] We shall discuss the reasons for this agreement later.

We consider now the general case. If in every successive φ_k we again set the coefficient of the term in $1/\xi$ equal to zero, we are led to the following expansions for E and z (given to the sixth approximation):

$$E = \omega_e(v + {}^1/_2) + B_e u\{1 + 6\alpha(v + {}^1/_2) + \alpha^2[30(v + {}^1/_2)^2 + 15/2 - 4u] + \alpha^3[140(v + {}^1/_2)^3 + 175(v + {}^1/_2) - 114u(v + {}^1/_2)] + \alpha^4[315\{2(v + {}^1/_2)^4 + 7(v + {}^1/_2)^2 + 9/8\} - 1560(v + {}^1/_2)^2 u - 326u + 48u^2]\} \tag{13}$$

$$z = y(1+\xi)^{\lambda-v}\exp -\alpha u\Big\{\frac{1}{1+\xi}\ [1+\alpha(2v+3)+\alpha^2(6v^2+22v$$
$$+15-4u)+\alpha^3(20v^3+126v^2+196v+105-57u-66uv)$$
$$+2\alpha^4(35v^4+236v^3+560v^2+475v-324uv^2-448uv-128u$$
$$+24u^2)]+\frac{\alpha}{(1+\xi)^2}\ [1+\alpha(4v+6-3u/2)+\alpha^2(16v^2+56v$$
$$+45-15uv-47u/2)+\alpha^3(64v^3+376v^2+688v+420$$
$$-105uv^2-385uv-217u/2+20u^2)]+\frac{\alpha^2}{(1+\xi)^3}\ [2-u/3$$
$$+\alpha(12v+20-2uv-9u)+\alpha^2(60v^2+236v+210-28uv^2/3$$
$$-320uv/3-428u/3+8u^2)]+\frac{\alpha^3}{(1+\xi)^4}\ [6-2u+\alpha(43v$$
$$+90-16uv-53u+5u^2/2)]+\frac{2\alpha^4}{(1+\xi)^5}\ (12-28u/5+u^2/5) \quad (14)$$
$$+\frac{4\alpha v}{\xi}\ [1+2\alpha(1+3v)+3\alpha^2(10v^2+7v+7-4u)+4\alpha^3(35v^3$$
$$+38v^2+92v+27-19u-57uv)]-\frac{\alpha^2 v}{\xi^2}\{(v-1)[3+20\alpha v$$
$$+7\alpha^2(15v^2+v+15)]-u\alpha[8+\alpha(171v-43)]\}+\frac{4\alpha^2 v(v-1)}{\xi^3}\ [1$$
$$+2\alpha(10v+1)/3+\alpha^2(35v^2+9v+25-18u)]-\frac{\alpha^3 v(v-1)}{\xi^4}\{3(2v$$
$$-3)+\alpha[5(17v^2-29v+6)/2-24u]\}+\frac{4\alpha^3 v(v-1)}{\xi^5}\ [2v$$
$$-3+2\alpha(7v^2-11v+1)]-\frac{3\alpha^4 v(v-1)}{\xi^6}\ (5v^2-17v+15)$$
$$+\frac{4\alpha^4 v(v-1)}{\xi^7}\ (5v^2-17v+15)\Big\}.$$

The value of λ checks Dunham but there are certain obvious difficulties connected with the characteristic function which arise at $\xi = 0$. In every order of approximation the highest power of $1/\xi$ in the exponent is always odd. Consequently, as ξ approaches 0 from the positive side, z tends to 0 but jumps to ∞ as ξ passes from $+0$ to -0, an indication that our assumed expansion does not converge at this point. We see, however, that when $v = 1$ and the approximation is sufficiently high, i.e., going beyond φ_2, the coefficients of all high powers of $1/\xi$ vanish and the leading term is an inverse square power of ξ. It has a negative coefficient so that $z = 0$ when $\xi = 0$. This behavior of the function for $v = 1$ is not destroyed at higher approximations. The form of the recurrence formulas for the φ_k shows that in this case the highest power of $1/\xi$ in the exponent will

be even for all orders and have a negative coefficient. For $v \geqslant 2$ the discontinuity at $\xi = 0$ persists at all orders of approximation. We carried the calculations for the coefficient of the highest power of $1/\xi$ up to the ninth order when it became evident that it cannot possibly vanish. The discontinuity at $\xi = 0$ does not invalidate the determination of the energy values; the difficulty in the eigenfunction would disappear if its complete expansion were known and the summation performed in the region where the series converges.

We have also evaluated the energy terms for a rotating anharmonic oscillator with a potential function of the form $V = b_0\xi^2(1 + b_1\xi)^2$. The perfect square form greatly simplifies the calculation. The results obtained to the third order again confirm Dunham's formulas.

This agreement may at first appear rather puzzling since we did not expand the rotational term. The use of the expansion transforms equation (3) to the form:

$$\frac{d^2y}{d\xi^2} - \frac{\xi}{\alpha}\frac{dy}{d\xi} + \left[\frac{\lambda}{\alpha} - u\sum_{k=0} C_k\xi^k\right]y = 0, \; C_k = (-1)^k(k+1),$$

where the summation over k extends only over a finite number of terms. If we solve this equation by our method we find that no term higher than C_{2k} will contribute to λ_k. Thus even though the expansion of the term in $1/(1 + \xi)^2$ diverges, it leads to correct values for the energy if a sufficient number of terms are kept.

The main advantage of this method is that it gives high order approximations to the eigenvalues with a comparatively small amount of labor. The eigenfunctions for which the expansion converges are also determined quite easily. We should like to point out now the difference between this and the W. K. B. method. Although there is a certain formal analogy in writing the solution in exponential form, the present treatment does not make use of complex integration. Moreover, our energy level conditions are not expressed in the form of phase integrals.

We plan to extend this procedure to the solution of other differential equations and also to present a discussion of its relationship to the W. K. B. and other methods.

In conclusion we want to thank Professor O. Laporte, Professor E. C. Kemble and Mr. Julian Schwinger for helpful discussions of the problem.

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[1] E. Fues, *Ann. Phys.*, **80**, 367 (1926).
[2] E. Fues, *Ibid.*, **81**, 281 (1926).
[3] A. Sommerfeld, *Wave-Mechanics*, Chap. I, §3D.
[4] J. L. Dunham, *Phys. Rev.*, **41**, 721 (1932).
[5] A. H. Wilson, *Proc. Roy. Soc.*, **118**, 617 (1928).

[6] When the boundary values of the variable are finite, it is not sufficient that the square of the absolute value of the eigenfunction should be integrable. The function itself must vanish at the limits. The function y and the integer v are characteristic of the equation in the interval from $-\infty$ to $+\infty$. Since physical significance can only be attached to the interval $-1 \leqq \xi < +\infty$, they are not the exact solutions of the physical problem. The correct eigenfunction would have the form $F = \bar{y} \exp(-\xi^2/4\alpha)$ with the analytic expression for $\bar{y}$ quite different from that for y, since $\bar{y} = 0$ at $\xi = -1$. However, because of the presence of the exponential factor in F even our inexact function involving y is very small at that point, namely, of the order of $\exp(-1/4\alpha)$, or only about e^{-25} times its equilibrium value. The corrections to the eigenvalues are of a correspondingly small order, i.e., very much smaller than any effects considered in this paper. Cf. the discussion in Pauling & Wilson, *Introduction to Quantum Mechanics*, Chap. X §35c, pp. 267–271.

[7] The summation extends to $v/2$ when v is even and to $\frac{1}{2}(v-1)$ when v is odd.

[8] It is to be stressed that the integral is an indefinite one.

[9] The separation of certain expressions into partial fractions is an essential part of the calculations. The following formulas have been used throughout:

$$\frac{1}{\xi(1+\xi)^n} = \frac{1}{\xi} - \sum_{k=1}^{n} \frac{1}{(1+\xi)^k}; \quad \frac{1}{\xi^2(1+\xi)^n} = \frac{1}{\xi^2} - \frac{n}{\xi} + \sum_{k=1}^{n} \frac{n-k+1}{(1+\xi)^k};$$

$$\frac{1}{\xi^3(1+\xi)^n} = \frac{1}{\xi^3} - \frac{n}{\xi^2} + \frac{n(n+1)}{2\xi} - \sum_{k=1}^{n} \frac{(n-k+1)(n-k+2)}{2(1+\xi)^k}$$

$$\frac{1}{\xi^n(1+\xi)} = \sum_{k=0}^{n-1} \frac{(-1)^k}{\xi^{n-k}} + \frac{(-1)^n}{1+\xi}$$

$$\frac{1}{\xi^n(1+\xi)^2} = \sum_{k=0}^{n-1} \frac{(-1)^k(k+1)}{\xi^{n-k}} + \frac{(-1)^n n}{1+\xi} + \frac{(-1)^n}{(1+\xi)^2}$$

$$\frac{1}{\xi^n(1+\xi)^3} = \sum_{k=0}^{n-1} \frac{(-1)^k(k+1)(k+2)}{2\xi^{n-k}} + \frac{(-1)^n n(n+1)}{2(1+\xi)} + \frac{(-1)^n n}{(1+\xi)^2} + \frac{(-1)^n}{(1+\xi)^3}.$$

[10] Whittaker and Watson, *Modern Analysis*, 4th ed., pp. 204–205 (1927).

THE EFFECT OF RADIOACTIVE PHOSPHORUS UPON THE BLOOD OF GROWING CHICKS

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Introduction.—The recent developments in the field of radiation which are making it possible artificially to prepare certain elements in the radioactive state are of very great significance to the biologist and physician. For entirely new technical procedures may be evolved, utilizing such elements, and representing definite improvement upon and extension of the present standard methods which require the use of x-rays and radium. Since

radioactive elements are identical chemically with their non-active isotopes, they react in biological systems in exactly the same manner.

Of these elements radioactive phosphorus seems to possess particular advantages both from the theoretical and practical points of view. Phosphorus is, of course, present in large amounts in the skeletal structure of animals and hence any effects observed as a result of administering the element in a radioactive form should be of value in the study of the physiology and pathology of bone. Phosphorus also occurs as a necessary constituent of the proteins, lipids and carbohydrates which are concerned with fundamental protoplasmic structure and with many of the more important vital reaction systems.

The value of radioactive phosphorus to the biologist is further enhanced by its comparatively long life (the half life is 14.5 days). This ensures to the investigator adequate time to prepare suitable compounds containing radioactive phosphorus, to administer them to large animals and to observe their effects over periods of relatively long duration. Finally the substance can be prepared in fairly large amounts by means of the cyclotron of Lawrence and Cooksey.[1]

Briefly the theory of this procedure is that when ${}_{15}P^{31}$ is bombarded in the cyclotron with deutrons (${}_{1}H^{2}$) phosphorus ${}_{15}P^{32}$ is formed together with a proton (${}_{1}H^{1}$). The latter form of phosphorus is unstable and decomposes into sulfur ${}_{16}S^{32}$ with the liberation of an electron (${}_{-1}e^{\circ}$). Hence it is this unstable isotope of phosphorus which produces beta rays or the radiation which would be responsible for any biological effects within the animal.

The first biological experiments using radioactive phosphorus were those of Chiewitz and Hevesy[2] who found that when the substance was administered to a white rat a large part of it was deposited in the bones. These investigators apparently fed very small amounts, which were insufficient to produce biological effects. Their findings suggested that when animals are fed radioactive phosphorus there may as a result be a selective irradiation in the region of heavy deposition, that is to say, the bone marrow. Now this is a portion of the animal's body which is exceedingly difficult to reach when the radiation (such as x-rays) is received by the external surface or skin and must then penetrate thick layers of tissue and bone. Hence, with radioactive phosphorus, one would pay particular attention to any disturbances in function on the part of the bone marrow. The latter, of course, would be reflected in distortions of the animal's blood picture, especially the polymorphonuclear leucocytes. There might well be a specific effect exerted on the polymorphonuclear leucocytes rather than on the lymphocytes, which are relatively more affected when animals are treated with x-rays.

In order to test these ideas, a series of experiments was performed in which radioactive phosphorus was administered to growing chicks. These

TABLE 1

RADIOACTIVE PHOSPHORUS

AGE OF BIRDS WHEN RADIOACTIVE PHOSPHORUS SAMPLES WERE FED	ACTIVITY OF PHOSPHORUS IN BIRDS IN MICROCURIES	RATIO: ACTIVITY IN MICROCURIES / WEIGHT OF BIRDS IN GRAMS		AGE IN DAYS	LEUCOCYTES NO./ CU. MM.	LYMPHOCYTES NO./ CU. MM.	POLYMORPHONUCLEARS NO./ CU. MM.	BASOPHILES NO./ CU. MM.	MONOCYTES NO./ CU. MM.	EOSINOPHILES NO./ CU. MM.	HEMOGLOBIN 16 GM./ 100 CC. BLOOD = 100 PER CENT
9–11 days	7	7/80	0.087								
				25	38,800	31,000	4,350	2,640	660	78	62
29–31 days	42	42/175	0.004	32	23,500	17,800	2,360	1,390	1,030	750	66
				37	23,300	17,250	3,350	1,400	1,235	0	66.4
				44	27,400	21,500	1,580	1,120	1,730	1,560	70.2
50–53 days	42	42/425	0.099	50	32,800	25,700	1,400	1,170	1,975	2,200	72.6
				57	31,800	25,000	1,560	890	1,530	2,670	64
				58	32,600	25,200	2,300	1,085	1,470	1,920	62.8
				65	35,400	31,000	920	950	1,100	1,900	67
72–73 days	40	40/775	0.052	72	36,000	29,000	2,000	1,100	2,200	1,100	71.4
				80	35,000	28,000	3,000	1,200	1,900	350	71
				83	34,000	28,000	4,000	700	1,000	750	73
80–81 days	190	190/875	0.22	84	33,000	25,000	5,000	1,100	1,700	0	70
				87	34,000	29,000	3,500	800	1,400	350	68
				94	32,000	28,000	1,800	900	650	450	74
				101	35,000	29,000	3,100	700	1,750	0	78
				109	32,000	27,000	3,200	900	1,300	0	85
Averages					32,000	26,000	2,700	1,130	1,400	900	71.5

NOTE: With respect to blood measurements each figure in tables 1 and 2 is the average of the values found on all five experimental or control birds at the age given. The averages listed in tables 1 and 2 are derived from all measurements at all ages, or a total of 80 determinations taken from five birds.

animals were chosen because of the authors' previous extended study of the blood pictures in chickens.

Procedure and Methods.—White leghorn chicks were selected as experimental animals. Because of their rapid growth, relatively large quantities of phosphorus are deposited in the bones. This should permit fairly complete deposition of the radioactive samples when administered.

Radioactive phosphorus was fed to one group of five birds by mixing it in their diet as phosphoric acid at the intervals stated in table 1. The radioactive phosphoric acid was prepared from red phosphorus, which had been bombarded in the cyclotron. The phosphoric acid was prepared by boiling the active red phosphorus in nitric acid (sp. g. 1.23). After the red phosphorus was converted to phosphoric acid the solution was evaporated to dryness in a steam bath to remove the excess nitric acid. The residue was then dissolved in water and treated with H_2S to remove traces of copper and other metallic salts. The precipitate was filtered off and discarded, and the remaining solution was made up to 500 cc. Of this $^1/_{500}$ was taken for an electroscopic analysis in order to determine the activity. The remainder was mixed with 300 gm. of the ration normally fed, dried on a steam bath and fed to the chicks.

Ordinary red phosphorus was fed to the control group as phosphoric acid prepared in the same manner. The diet given the chicks was composed of:

Wheat	20 parts
Corn	55.5 parts
Dried blood meal	3 parts
Casein	10 parts
Alfalfa leaf meal	5 parts
Calcium carbonate	1 part
Sodium chloride	0.5 part
Dried brewer's yeast	3 parts
Sardine oil	2 parts

The usual supplement of 1 per cent calcium phosphate was purposely omitted from the diet to enhance the uptake of the radioactive samples of phosphorus when given.

Samples of blood were taken from the control group and the radioactive group at stated intervals for 109 days (see tables 1 and 2). The hemoglobin, leucocytes, lymphocytes, polymorphonuclear leucocytes, basophiles, eosinophiles and monocytes were estimated. The methods may be found in previously published work.[3]

Results and Discussion.—The effect of radioactive phosphorus upon the blood constituents of the experimental animals was found to differ in certain aspects from the effect observed when animals are irradiated with x-rays. In particular the lymphocytes of the chicks which received the radioactive

phosphorus were practically unaffected, whereas their number is usually decreased by x-radiation.[4] This difference, as was suggested in the introduction, is of considerable significance since it indicates that a more or less selective irradiation is possible when such elements as radioactive phosphorus are introduced internally by some such method as was here used. There were two periods during the experiment when the number of lymphocytes decreased slightly for a short time. These decreases occurred during and immediately after the feeding of large doses of radioactive phosphorus (after the 29th and 80th days—see table 1). At these times the radioactivity in comparison to the weight of the birds reached its maxima for the experimental period. It will be noted, however, that the number of lymphocytes quickly returned to normal, although the activity of the phosphorus was (minus the decay) approximately

TABLE 2

CONTROLS

AGE IN DAYS	LEUCOCYTES	LYMPHOCYTES	POLYMORPHONUCLEARS	BASOPHILES	MONOCYTES	EOSINOPHILES	HEMOGLOBIN 16 GM./100 CC. BLOOD = 100 PER CENT
	NO./CU.MM.	NO./CU.MM.	NO./CU.MM.	NO./CU.MM.	NO./CU.MM.	NO./CU.MM.	
25	28,000	20,700	5,500	820	1,000	0	58.2
32	27,000	23,035	4,050	650	365	0	66
37	34,000	27,500	3,950	950	1,835	0	61
44	29,200	22,400	4,760	560	1,340	134	69
50	30,800	24,700	3,500	990	1,350	215	66.2
57	35,000	26,000	5,600	1,540	1,785	140	62
65	30,400	26,000	2,500	550	730	850	63
72	36,000	28,000	4,000	1,300	1,800	220	65.6
80	39,000	32,000	4,500	1,000	1,700	150	70
87	38,000	31,000	4,000	1,000	2,000	200	68
94	40,000	34,000	3,500	1,000	1,600	160	71
101	39,000	32,000	4,500	800	1,500	0	75
109	35,000	30,000	3,500	700	1,400	0	76
			Averages				
	34,000	27,500	4,100	910	1,400	160	68

the same as when the effect was initially observed. A reasonable explanation for these observations is that a large portion of the radioactive phosphorus ingested had not as yet been deposited, but was still freely circulating in the blood and other body fluids where lymphocytes are to be found in large numbers. As the phosphorus was progressively withdrawn from the circulating media for deposition, its direction effect on the lymphocytes would diminish and the lymphocyte count would rise correspondingly. In addition to direct action on circulating lymphocytes, radioactive phosphorus dissolved in the body fluids would be freely accessible to the tissue which generates the lymphocytes and therefore might, to some extent, suppress the formation of new cells.

In this connection the effect of x-rays was checked by subjecting three birds to x-radiation (300 r). These birds had been grown under conditions otherwise identical with those which were given radioactive phosphorus. The result was a distinct decrease in lymphocytes, a generally characteristic x-ray effect.

A difference in the effect on polymorphonuclear leucocytes was also observed. When the three birds, mentioned above, were subjected to x-radiation there was first a preliminary increase which is likewise characteristically induced by x-rays. Quickly thereafter, however, the number returned to substantially the normal (see table 3).

On the other hand, when birds were given radioactive phosphorus a profound decrease in the polymorphonuclear leucocytes was the result. The effect on these cells was approximately proportional to the effective dose of phosphorus. The latter may be considered as the ratio:

$$\frac{\text{activity of phosphorus absorbed, in microcuries}}{\text{weight of birds in grams}}$$

TABLE 3

BLOOD PICTURE AVERAGES OF THREE BIRDS BEFORE AND AFTER TREATMENT WITH X-RAY (DOSE APPROXIMATELY 300 r)

DATE	LEUCOCYTES	LYMPHOCYTES		POLYMORPHONUCLEARS		BASOPHILES		MONOCYTES		EOSINOPHILES		HEMOGLOBIN 16 GM./100 CC. BLOOD = 100 PER CENT
		PER CENT	NO./CU. MM.	PER CENT	NO./CU. MM.	PER CENT	NO./CU. MM.	PER CENT	NO./CU. MM.	PER CENT	NO./CU. MM.	
1/27/37	47,000	86	41,000	9	4,200	1	470	4	1,800	0	0	66
2/19/37	47,000	85	40,000	9	4,200	2	900	4	1,800	0	0	75
GIVEN X-RAY TREATMENT 2/20/37												
14 hrs. after	18,000	32	5,700	62	11,000	0	0	6	1,000	0	0	76
38 hrs. after	12,000	55	6,600	40	4,800	0	0	5	600	0	0	79
62 hrs. after	13,000	61	8,000	35	4,500	1	130	3	390	0	0	81
110 hrs. after	12,000	66	8,000	25	3,000	1	100	5	600	3	300	80

Since the birds were growing rapidly during the experiment, it is convenient to express the dosage in terms of this ratio as has been done in presenting the data in table 1. Furthermore, it is thus possible to compare the amount of radiation received by the birds during the course of the experiment in terms of unit weight.

From the data in table 1 it will also be observed that whenever a dose of radioactive phosphorus was administered there was a sharp decrease in polymorphonuclear leucocytes. But as the activity of a given sample grew weaker due to natural decay the number of these cells increased at a rate more or less proportional to the rate of decay.

The evidence from polymorphonuclear leucocytes therefore points in the same direction as that from the lymphocytes. That is, the effect in-

duced by radioactive phosphorus on the former cells is apparently the result of the more or less selective deposition of the element in bone. As a consequence the bone marrow, the site of formation of polymorphonuclears, receives relatively much more irradiation than the other tissues of the body, a condition which could not arise were the radiation to come from a source external to the animal. These observations suggest the feasibility of utilizing radioactive phosphorus for the purpose of investigating myelogenous leukemia and other hematopoietic disturbances.

In addition to the effects described above, an increase in eosinophiles was noted. Similar findings have been reported when guinea pigs are treated with x-rays.[5] The basophiles and monocytes did not appear to be much affected. The hemoglobin per cent of the birds fed radioactive phosphorus was for the most part higher than the controls, although we have at present no explanation to offer for this observation. The growth of both groups was normal, and except for a few days after the largest dose was given (81st day) when animals developed temporary diarrhea, there was no evidence of illness. The blood pictures of the control group were similar to those previously described by us and Englebreth-Holm.[3,6]

Summary.—1. Because of the importance of phosphorus to living animals, its artificially prepared radioactive isotope ($_{15}P^{32}$) is a valuable tool in biological investigations.

2. The administration of radioactive phosphoric acid (birds were used as experimental animals) gives rise to effects upon the constituents of the blood that are not obtained with x-rays.

3. The lymphocytes which are sensitive to x-irradiation were not particularly affected.

4. The polymorphonuclear leucocytes were greatly decreased in numbers after the administration of radioactive phosphorus.

5. This specific effect upon polymorphonuclear leucocytes is attributed to the selective deposition of radioactive phosphorus in the bones, which allows bombardment of the bone marrow with beta rays.

6. The basophiles and monocytes were not appreciably affected, although an increase in eosinophiles was noted.

7. Birds treated in this manner grew normally and any permanent ill effects resulting from the administration of the radioactive phosphorus were not observed.

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Laboratory, and especially Mr. Philip Abelson, who determined the activity of the various phosphorus samples.

We appreciate the interest of Dr. John Lawrence and Mr. Paul C. Abersold and their assistance with x-ray exposures.

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EFFECT OF THE ROOTS ON THE PRODUCTION OF AUXIN BY THE COLEOPTILE

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Since it had been shown by Skoog (1936, 1937) that removal of the seed (endosperm and scutellum) of Avena seedlings greatly reduces the production of auxin in the coleoptile tip, the question arose whether removal of the root system would have any effect upon the auxin relations of the plant. This was the purpose of the present investigation.

From seedlings of "Victory oats" which were grown in physiological darkrooms under standard conditions (see, e.g., Went 1935), the root system was cut off with a sharp razor blade. At the same time the coleoptile was decapitated by severing 2 mm. of the tip. In order to allow water uptake the level of the water in the zinc trays, from which the plants ordinarily take up their water, was raised until the endosperm was half submerged. The level of the water of the trays in which the roots of the intact control plants were hanging was also raised to the same extent. The plants used were of the same age as required for the standard auxin tests. Three hours after the first decapitation the end of the coleoptiles was cut off again, the primary leaf pulled loose and agar blocks containing indole-3-acetic acid (hetero-auxin) were unilaterally applied to the top of the coleoptile. The resulting curvatures were photographed 110 minutes later. The results are given in table 1 from which it is clear that removal of the

roots, 3 hours before the application of auxin, does not change the response to auxin of the coleoptiles. However, if the period of time between the removal of the roots and the application of auxin was 15 to 20 hours instead of 3, the sensitivity was markedly increased in regard to plants with an intact root system. Table 2 shows that this increase in sensitivity may

TABLE 1

Auxin curvatures of plants with and without their root system. Roots removed 3 hours before the auxin was applied. Curvatures photographed 110 minutes after the auxin was applied. Average values of 12 plants.

EXPERIMENT NUMBER	AUXIN CONCENTRATION	AUXIN CURVATURES WITH ROOTS	AUXIN CURVATURES WITHOUT ROOTS
61105	0.374 mg. per l.	10.2	12.1
		10.3	10.6
61109	0.187 mg. per l.	5.5	3.6
	0.374 mg. per l.	10.3	11.9
	0.561 mg. per l.	15.7	17.2

TABLE 2

Auxin curvatures of plants with and without their root system. Roots removed 15 hours before the auxin was applied. Curves photographed 90 minutes after the auxin application. Averages of 12 plants.

EXPERIMENT NUMBER	AUXIN CONCENTRATION	AUXIN CURVATURES WITH ROOTS	AUXIN CURVATURES WITHOUT ROOTS
61113	0.187 mg. per l.	4.6	10.3
	0.374 mg. per l.	12.8	18.7
61210	0.117 mg. per l.	22.8	21.1
		19.3	21.2
	0.059 mg. per l.	7.1	11.2
		8.9	11.4

be more than 100%. Especially if the curvatures are far below the maximum angle, this effect is prominent. Curvatures near the maximum angle (about 20°) do not show a difference between plants with and without their root system.

It has been pointed out repeatedly (Van der Wey 1931, van Overbeek 1936 and others) that the less auxin the coleoptiles contain the more sensitive they are to applied auxin. Hence, in order to explain the higher sensitivity of the plants from which the roots were removed, it was assumed that the production of growth hormone by the coleoptile tip was reduced as compared with plants with normal root systems. This was actually observed (Fig. 1). The tips of plants from which the roots had been removed for about 15 hours were placed on agar blocks. After the tips had been standing for 2 hours the hormone content of those blocks was determined in the usual way. The blocks on which the tips of intact

plants had been standing had a relative auxin content of 11.3° (Fig. 1), whereas the blocks on which the tips of plants from which the roots had been removed had been standing had only a concentration of 6.5°. In the same experiment the auxin production of plants from which the seed had been removed (roots intact) was determined, and also of plants from which both roots and seed were removed (cut off at the base of the mesocotyl). Figure 1 also shows the length of the plants of the various groups. It is clear that the more the production of auxin was inhibited the less the

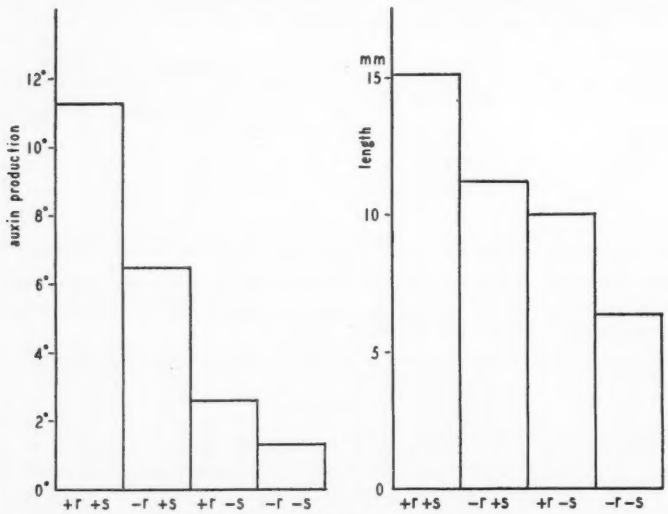

FIGURE 1

Auxin production and length of coleoptiles of Avena seedlings which were left intact (+ r + s) or from which the roots (− z + s), the seed (+ z − s) or both were removed. Removals were made 20 hours before the determination of the auxin production. The length represents that part of the coleoptile which sticks out above the glass holder. Experiment number 61222.

plants had grown. So it seems that the auxin production, in these four groups of plants, determines their growth in length. Auxin curvatures of intact Avena plants increase with time until about $1^1/_2$ to 2 hours after the application of auxin. Then they decrease gradually, which is shown to be connected with new formation of auxin ("regeneration") by the decapitated plant (see Skoog 1937). Skoog showed that deseeded plants almost lack this power to produce auxin after decapitation. Hence, the auxin curvature of such plants increases over a much longer period of time. Plants from which the root system had been removed showed regeneration as is shown in table 3. Here it can be seen that, 5 hours after the auxin

blocks had been put on, the curvatures in the plants without roots are smaller than they were $3^1/_2$ hours before. This going back of the curvatures shows that regeneration takes place. The plants without seed but with roots also show some regeneration but at a later stage than the intact or the rootless plants. The plants which have been cut off at their base and placed on wet cotton do not show any sign of regeneration; their curvature

TABLE 3

Auxin curvatures of plants from which either the seed, the root system or both were removed, at various times after auxin application. Removals were made 15 hours before the auxin was applied. Averages of 24 plants. Auxin concentration 0.059 mg. per l. Experiment 61222.

	TIME AFTER AUXIN APPLICATION:		
	90 MINUTES	5 HOURS	8 HOURS
Seed and roots present	10.3	5.3	straight
Roots removed	14.9	11.2	gone back
Seed removed	8.7	16.4	gone back
Seed and roots removed	6.7	19.7	about 30° curvature

increases steadily. So it seems that such plants are even more suited to measure minute amounts of auxin than the deseeded plants of a test developed by Skoog.

Conclusions.—The fact that deseeded plants do not produce as much auxin as intact plants do, has been explained by Skoog by assuming that removal of the seed also removes the source from which the precursor of auxin is coming. A similar explanation may hold for the plants which were cut off at their base. According to Skoog this inactive precursor is present in the seed and is polarly transported in apical direction, principally in the same way as auxin is transported in basal direction. In derooted plants, however, the endosperm is retained, so the precursor may go up into the shoot and there be activated into auxin. The fact that regeneration takes place in derooted plants (with their auxin source intact) and practically not in deseeded plants is in agreement with the above view. If, however, all the active auxin in the shoot should derive from the precursor only and this precursor should be transported only by a mechanism of polar transport as described above, it is hard to see why the auxin production of the derooted plants is so much less than that of the intact plants.[1] The root system, which is the only factor by which the derooted and the intact plants differ, must have a part in the apical transport of auxin or its precursor.

It has been shown by Kögl, Erxleben and Haagen Smit (1934), Laibach and Meyer (1935) and others that germinating seeds contain active auxin. Hitchcock and Zimmerman (1935) showed that auxin when taken up by the roots can be transported toward the shoot by the transpiration stream. So Avery and Burkholder (1936) assumed that the active auxin may move upward from the seed through the vascular bundles to the coleoptile tip

from where it is dispersed downward. This conclusion first seemed unacceptable in the light of Van der Wey's investigations which showed that in *sections* of coleoptiles the auxin can be transported in basal direction and not in apical direction. However, if in the Avena seedling the roots are actively engaged in the upward transport of auxin as the experiments presented in this paper seem to indicate, Avery and Burkholder's opinion is not necessarily in conflict with Van der Wey's experiments.

As a tentative mechanism of the upward transport of auxin in the Avena seedling I would suggest combining the above-mentioned opinions in the following way. Auxin is present in the seed in two forms: the active form and the inactive precursor. Both forms may be transported from the seed toward the apex of the shoot, but the active form goes up only with the transpiration- or root-pressure stream.

Summary.—Removal of the root system for a period of 15 to 20 hours reduces the auxin production of the coleoptile tip of Avena seedlings markedly. This reduced production causes a decreased growth, but at the same time an increased sensitivity for auxin. This increase in sensitivity may be as high as 100%. New formation of auxin after decapitation takes place in plants from which the roots have been removed.

Plants from which both roots and seed have been removed have a lower initial sensitivity but they do not "regenerate," hence their curvatures do not decrease but increase steadily. This makes such plants even more useful to detect small amounts of auxin than the deseeded plants.

It is suggested that there are two ways by which the auxin from the seed reaches the apex of the shoot, viz., as inactive precursor which may be transported without the presence of the root system, and as active auxin which may go up with the transpiration- or root-pressure stream.

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[1] It may be argued that the wound made by removing the roots is a principal factor. This is not likely, however, since wounding of the mesocotyl (by scraping of the epidermis) of intact plants hardly affects the growth in length. That the decrease of auxin production is not due to a loss of auxin through the wound made by removing the roots was also shown by plugging up the wound.

THE IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP

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In the present note I state certain theorems on the irreducible representations of the symmetric group in the hope that they may lighten the labors of those carrying out theoretical investigations in nuclear physics. A detailed discussion of these results will, it is planned, appear later in the *American Journal of Mathematics*.

THEOREM 1. Denoting by $\{\lambda_1, \lambda_2, \ldots \lambda_\kappa\}$ the aggregate of characters of that irreducible representation $D(\lambda_1, \lambda_2, \ldots \lambda_\kappa)$ of the symmetric group on n letters which is associated with the partition $\lambda_1 + \lambda_2 + \ldots + \lambda_\kappa = n$ of n, $(\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_\kappa > 0)$ and by $\alpha = (\alpha_1, \ldots \alpha_n)$ the class containing α_1 unary cycles, α_2 binary cycles, α_3 ternary cycles and so on $(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + n\alpha_n = n)$ then the character $\{\lambda_1, \ldots \lambda_\kappa\}_\alpha$ of $D(\lambda_1, \ldots \lambda_\kappa)$ corresponding to any class α which contains *at least* one cycle on p letters $(1 \leqslant p < n)$ may be obtained from the characters of the symmetric group on $n - p$ letters by means of the formula

$$\{\lambda_1, \lambda_2, \ldots \lambda_\kappa\}_\alpha = \{\lambda_1 - p, \lambda_2, \ldots \lambda_\kappa\}_{\alpha'} + \{\lambda_1, \lambda_2 - p, \lambda_3, \ldots \lambda_\kappa\}_{\alpha'} + \ldots + \{\lambda_1, \ldots \lambda_\kappa - p\}_{\alpha'}$$

where α' is the class of the symmetric group on $n - p$ letters containing one less cycle on p letters than the corresponding class α of the symmetric group on n letters. In using this formula it is understood that when the subtraction of p forces a reversal in the normal descending order this normal order is restored by interchanging the two elements which are in reversed order, increasing the one which was on the left by unity and decreasing the one which was on the right by unity, and prefixing a minus sign to the $\{\ldots\}$. E.g., $\{3, 5, 2\} = -\{4, 4, 2\}$. It is an immediate consequence that $\{\ldots a, b, \ldots\} = 0$ if $b = a + 1$ since $\{\ldots a, b, \ldots\} = -\{\ldots b - 1, a + 1, \ldots\} = -\{\ldots a, b, \ldots\}$ implying $\{\ldots a, b, \ldots\} = 0$. Similarly $\{\ldots a, b, c, d, \ldots\} = 0$ if $c = a + 2$ or if $d = a + 3$ and so on. Furthermore zeros occurring at the end are dropped and any $\{\ldots\}$ containing a negative number at the end is dropped. The particular case of this theorem when $p = 1$ is due to Schur and has been the most useful in the construction of the character tables of the symmetric group. In the complete form of statement all characters of the symmetric group on n letters may be calculated from the, supposed known, characters of the symmetric groups on $m < n$ letters save the characters of the class con-

taining one cycle on n letters. Fortunately for the force of the theorem these characters are trivially evident; all being zero save $\{n-k, 1, 1, \ldots 1\}$ which equals $(-1)^k$, $k = 0, 1, \ldots n-1$. Furthermore a repeated application of the theorem reduces the determination of the characters of a class for which $\alpha_1 \neq 0$ to that of the dimensions (= characters of the unit class) of representations of the symmetric group on α_1 letters; and for these the convenient formula of Frobenius is available. E.g., suppose we wish to calculate the characters of the symmetric group on $n = 20$ letters corresponding to the class containing $\alpha_1 = 12$ unary cycles and $\alpha_8 = 1$ cycle on 8 letters. As an illustration consider the representation $D(9, 6, 3, 2)$. Applying our theorem with $p = 8$ we obtain

$$\{9,6,3,2\}_\alpha = \{1,6,3,2\}_{\alpha'} + \{9,-2,3,2\}_{\alpha'} + \{9,6,-5,2\}_{\alpha'} + \{9,6,3,-6\}_{\alpha'}$$

where α' is the unit class (i.e., that one consisting of the unit permutation, which contains 12 unary cycles) of the symmetric group on 12 letters. The two terms at the end are dropped (the second from the end because $\{9, 6, -5, 2\} = -\{9, 6, 1, -4\} = 0$). The first term is dropped because the third member 3 of the partition is bigger by two units than the first member 1. The only remaining term, the second, transforms thus: $\{9, -2, 3, 2\} = -\{9, 2, -1, 2\} = \{9, 2, 1, 0\} = \{9, 2, 1\}$ and so the desired character is the dimension of the representation $D(9, 2, 1)$ of the symmetric group on 12 letters, namely, $12!\,10.\,8.\,2/11!\,3! = 320$.

THEOREM 2. We denote by $\Delta(\lambda_1, \lambda_2, \ldots \lambda_\kappa)$ the *reducible* representation of the symmetric group on n letters which is associated with the partition $(\lambda_1, \lambda_2, \ldots \lambda_\kappa)$ of n, i.e., the representation by means of the permutation matrices furnished by the cosets of that subgroup which permutes the first λ_1 letters, the second λ_2 letters, ... and so on amongst themselves. Then it is well known that $\Delta(\lambda_1, \lambda_2) = D(\lambda_1, \lambda_2) + D(\lambda_1 + 1, \lambda_2 - 1) + D(\lambda_1 + 2, \lambda_2 - 2) + \ldots$, there being λ_2 terms in the summation on the right. The present theorem extends this result from the case $k = 2$ to a general value of k. We shall merely state the theorem for $k = 3$, the general mode of procedure being then clear (we shall illustrate by means of an example for which $k = 4$). Denote by $\Sigma(\lambda_1, \lambda_2)$ the summation on the right of the expression for $\Delta(\lambda_1, \lambda_2)$ and by $(\mu, \Sigma(\lambda_1, \lambda_2))$ the summation $D(\mu, \lambda_1, \lambda_2) + D(\mu, \lambda_1 + 1, \lambda_2 - 1) + \ldots$ obtained by prefixing a μ to each partition in $\Sigma(\lambda_1, \lambda_2)$. Then

$$\Delta(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1, \Sigma(\lambda_2, \lambda_3)) + (\lambda_1 + 1, \Sigma(\lambda_2 - 1, \lambda_3)) + (\lambda_1 + 1, \Sigma(\lambda_2, \lambda_3 - 1)) + (\lambda_1 + 2, \Sigma(\lambda_2 - 2, \lambda_3)) + (\lambda_1 + 2, \Sigma(\lambda_2 - 1, \lambda_3 - 1)) + (\lambda_1 + 2, \Sigma(\lambda_2, \lambda_3 - 2)) + \ldots$$

Thus, for $n = 5$,

$$\begin{aligned}\Delta(2, 2, 1) &= D(2, 2, 1) + D(2, 3) + D(3, 1, 1) + D(3, 2) + D(3, 2) \\ &\quad + D(4, 1) + D(4, 1) + D(5) \\ &= D(2, 2, 1) + D(3, 1, 1) + 2D(3, 2) + 2D(4, 1) + D(5)\end{aligned}$$

the same provisions as to reversal of order still holding.

Similarly, for $n = 4$,

$$\Delta(2, 1, 1) = D(2, 1, 1) + D(2, 2) + 2D(3, 1) + D(4)$$

and, for $n = 3$,

$$\Delta(1, 1, 1) = D(1, 1, 1) + 2D(2, 1) + D(3).$$

As an example of the method when $k = 4$ consider the representation $\Delta(3, 2, 2, 1)$, of dimension 1680, of the symmetric group on 8 letters. For the first group of terms we simply prefix a 3 to the partitions in the formula, given above, for $\Delta(2, 2, 1)$. For the next group of terms, where we increase the first member 3 of the partition to 4, we have to deal with (2, 1, 1) twice and (2, 2) once (it being allowable to write all the derived partitions of $n - (\lambda_1 + 1) = 4$ in normal descending order). For the next group, in which we increase the 3 to 5 we have to deal with (2, 1) four times and (1, 1, 1) once. For the next group, in which 3 is increased to 6 we have to deal with (1, 1) three times and (2) twice; and then when the 3 is increased to 7 we have to deal with (1) three times. Finally, when the 3 is increased to 8 we have to deal with (0) once (bringing into light the fact that the identity representation is contained once in each and every reducible representation $\Delta(\lambda_1, \lambda_2, \ldots \lambda_\kappa)$). We obtain finally

$$\begin{aligned}\Delta(3, 2^2, 1) &= D(3,2^2,1) + D(3^2,1^2) + 2D(3^2,2) + 2D(4, 2, 1^2) + \\ &\quad 3D(4, 2^2) + 5D(4, 3, 1) + 2D(4^2) + D(5, 1^3) + 6D(5, 2, 1) + \\ &\quad 5D(5, 3) + 3D(6, 1^2) + 5D(6, 2) + 3D(7, 1) + D(8).\end{aligned}$$

An important consequence of the theorem is that there are no $D(\lambda_1, \lambda_2, \ldots \lambda_j)$ with $j > \kappa$ in $\Delta(\lambda_1, \lambda_2, \ldots \lambda_\kappa)$.

THEOREM 3. We shall denote by $D(\lambda_1, \lambda_2, \ldots \lambda_\kappa) \times D(\mu_1, \mu_2, \ldots \mu_j)$ the direct product of the indicated irreducible representations of the symmetric groups on $n = \lambda_1 + \lambda_2 + \ldots + \lambda_\kappa$ and $m = \mu_1 + \mu_2 + \ldots + \mu_j$ letters, respectively. This direct product is a representation (reducible) of the symmetric group on $n + m$ letters. The number of times various of these direct products contain the different irreducible representations of the symmetric group on $n + m$ letters is found as follows. Let us denote by $E_p(\lambda_1, \lambda_2, \ldots \lambda_\kappa)$ the sum

$$E_p(\lambda_1, \lambda_2, \ldots \lambda_\kappa) = D(\lambda_1 + p, \lambda_2, \ldots \lambda_\kappa) + \ldots + D(\lambda_1, \lambda_2, \ldots \lambda_\kappa + p)$$

by $E_{p,q}(\lambda_1, \lambda_2, \ldots \lambda_\kappa)$ the sum obtained by adding p to one member of the partition and q to another in all possible ways; thus

$$E_{p,q}(\lambda_1, \lambda_2, \lambda_3) = D(\lambda_1 + p, \lambda_2 + q, \lambda_3) + D(\lambda_1 + p, \lambda_2, \lambda_3 + q) + D(\lambda_1, \lambda_2 + p, \lambda_3 + q) + D(\lambda_1 + q, \lambda_2 + p, \lambda_3) + D(\lambda_1 + q, \lambda_2, \lambda_3 + p) + D(\lambda_1, \lambda_2 + p, \lambda_3 + q)$$

with the previous provisions as to reversals of normal descending order as before, and so on.

Then (*a*) $D(1) \times D(\lambda_1, \ldots \lambda_\kappa) = E_1(\lambda_1, \ldots \lambda_\kappa, 0)$. E.g., $D(1) \times D(2,1) = E_1(2, 1, 0) = D(2, 1, 1) + D(2, 2) + D(3, 1)$.

(*b*) $D(2) \times D(\lambda_1, \ldots \lambda_\kappa) = E_2(\lambda_1, \ldots \lambda_\kappa, 0) + E_{11}(\lambda_1, \ldots \lambda_\kappa, 0)$. E.g., $D(2) \times D(3, 1) = E_2(3, 1, 0) + E_{11}(3, 1, 0) = D(5, 1) + D(3, 3) + D(4,2) + D(4, 1, 1) + D(3, 2, 1)$.

(*c*) $D(3) \times D(\lambda_1, \ldots \lambda_\kappa) = E_3(\lambda_1, \ldots \lambda_\kappa, 0) + E_{21}(\lambda_1, \ldots \lambda_\kappa, 0) + E_{111}(\lambda_1, \ldots \lambda_\kappa, 0)$ and so on.

(*d*) $D(1, 1) \times D(\lambda_1, \ldots \lambda_\kappa) = E_{11}(\lambda_1, \ldots \lambda_\kappa, 0) + D(\lambda_1, \ldots \lambda_\kappa, 1, 1) = E_{11}(\lambda_1, \ldots \lambda_\kappa, 0, 0)$. E.g., $D(1, 1) \times D(3, 1) = E_{11}(3, 1, 0, 0) = D(3, 1^3) + D(3, 2, 1) + D(4, 1^2) + D(4, 2)$.

(*e*) $D(1, 1, 1) \times D(\lambda_1, \ldots \lambda_\kappa) = E_{111}(\lambda_1, \ldots \lambda_\kappa, 0, 0, 0)$.

(*f*) $D(2, 1) \times D(\lambda_1, \ldots \lambda_\kappa) = D(\lambda_1, \ldots \lambda_\kappa, 2, 1) + E_{21}(\lambda_1, \ldots \lambda_\kappa, 0) + E_{111}(\lambda_1, \ldots \lambda_\kappa, 0, 0)$.

It is hoped that these illustrations will point the method to the interested reader. The complete avoidance of the troublesome use of the character tables is very time saving.

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GROUPS OF ORDER LESS THAN 2^m HAVING $m - 1$ OR $m - 2$ INDEPENDENT GENERATORS

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A group whose order is less than 2^m can obviously not have as many as m independent generators. If it has $m - 1$ independent generators its order is at least as large as 2^{m-1}, and when it is of this order it is abelian and of type 1^{m-1}. If its order exceeds 2^{m-1} but is less than 2^m and it has $m - 1$ independent generators the number of prime factors in its order is $m - 1$ and hence its order is 3.2^{m-2}, $m > 1$. When $m = 2$ it is the group

of order 3, and when $m = 3$ it is one of the two groups of order 6. When it is the cyclic group of this order it has either one or two independent generators depending on the way in which this generator or these generators are chosen. It remains to determine the possible groups when $m > 3$. It is obvious that no one of the $m - 1$ independent generators could have an order which exceeds 3 and that there could not be more than one such generator of order 3. If one of the independent generators is of this order the group generated by it is an invariant subgroup of the entire group since the latter has $m - 1$ independent generators and its order is less than 2^m.

If none of the independent generators is of order 3 then at least two of them are non-commutative and hence they generate the non-cyclic group of order 6. As this must also be an invariant subgroup of the entire group it results that each of the possible groups has an invariant subgroup of order 3. When the operators of this subgroup are in the central of the group it is the direct product of the group of order 3 and the abelian group of order $2^{m-2}$and of type 1^{m-2}. When these operators are not in the central the group is the direct product of the non-cyclic group of order 6 and the abelian group of order 2^{m-3} and of type 1^{m-3}. Hence there results the following theorem: *When* $m > 2$ *there are three and only three groups of order less than* 2^m *which have separately as many as* $m - 1$ *independent generators, viz., the abelian group of order* 2^{m-1}*and of type* 1^{m-1}*, the direct product of the group of order* 3 *and the abelian group of order* 2^{m-2} *and of type* 1^{m-2}*, and the direct product of the non-cyclic group of order* 6 *and the abelian group of order* 2^{m-3} *and of type* 1^{m-3}.

If a group whose order is less than 2^m has $m - 2$ independent generators its order is at least 2^{m-2} and when it is of this order it is abelian and of type 1^{m-2}. According to the theorem of the preceding paragraph there are three and only three such groups whose orders are less than 2^{m-1}, $m > 3$. Under the same condition as regards the value of m the number of these groups of order 2^{m-1} is known to be $3(m-2)/2$ when m is even and $(3-7)/2$, when m is odd.[1] It remains therefore to determine these groups when their orders exceed 2^{m-1} but are less than 2^m. The $m - 2$ independent generators could obviously not contain more than three of order 3 and when they include three such generators, the group involves an invariant abelian subgroup of order 27 and of type 1^3, which is generated by them. This is a special case of the following elementary theorem: *If the number of independent generators of a group is equal to the number of prime factors which divide its order then its Sylow subgroup involving operators of largest prime order is invariant.*

The $m - 2$ independent generators of the groups under consideration could clearly not include one whose order exceeds 7 and when they include one of this order the group generated by it is invariant under the entire

group and each of the remaining $m - 3$ independent generators is of order 2, and every two of them are commutative. They therefore generate an abelian Sylow subgroup of order 2^{m-3} and of type 1^{m-3}. When $m = 3$ the group of order 7 is the only such group, but when $m > 3$ there are two and only two such groups for every value of m. One of these is the direct product of the group of order 7 and the abelian group of order 2^{m-3} and of type 1^{m-3}, while the other is the direct product of the dihedral group of order 14 and the abelian of order 2^{m-4} and of type 1^{m-4}. These two groups are therefore the only ones whose orders are less than 2^m, $m > 3$, which separately have $m - 2$ independent generators such that one of them is as large as 7. If one of the $m - 2$ independent generators is of order 6 the group contains also $m - 1$ independent generators and hence is included among those determined above.

Suppose that the $m - 2$ independent generators of a group G include one whose order is 5. The subgroup generated by it is invariant under G in view of the fact that the order of G is less than 2^m. If an additional independent generator is of order 3 then G contains invariantly the group of order 15. All of the remaining $m - 4$ independent generators of G are then of order 2 and every two of them are commutative. There are clearly four and only four such groups of order 30 since the group of isomorphisms of the group of order 15 is the abelian group of order 8 and of type 2, 1. The number of distinct groups of order 15.2^{m-4}, $m > 5$, is five since the number of the combinations of four things, taken two at a time is ten, when repetitions are allowed. That is, *there are five and only five distinct groups of order less than* 2^m *which separately contain* $m - 2$ *independent generators, including one of order* 5 *and one of order* 3, *whenever* $m > 5$. *When* $m = 5$ *the number of these groups is four.*

When one of the $m - 2$ independent generators of G is of order 5 and none of them is of order 3 two cases present themselves. In one of these two of the $m - 3$ remaining independent generators are non-commutative and in the other every two of these independent generators are commutative. In the former case the two non-commutative generators of order 2 generate the non-cyclic group of order 6 and hence the $m - 2$ independent generators could have been so selected as to contain one of order 5 and one of order 3. This therefore comes under the theorem noted in the preceding paragraph and requires no further consideration here. When every two of the $m - 3$ independent generators of order 2 are relatively commutative they generate the abelian Sylow subgroup of order 2^{m-3} and of type 1^{m-3}. Hence there are two and only two such groups of order 5.2^{m-3} for every value of $m > 3$. One of these is the direct product of the group of order 5 and the abelian group of order 2^{m-3} and of type 1^{m-3} while the other is the direct product of the dihedral group of order 10 and the abelian group of order 2^{m-4} and of type 1^{m-4}.

We have now considered all the possible groups when the order of at least one of the $m - 2$ independent generators of G is as large as 5. When the order of one and only one of these generators is 4 and the subgroup generated by it is invariant one and only one of the remaining independent generators can be of order 3. When there is such a second independent generator G involves the cyclic group of order 12 invariantly and the remaining $m - 4$ independent generators are of order 2 or of order 4. There are six and only six such groups of order 24 and there are eight and only eight of every order of the form 12.3^{m-4} for every value of $m > 5$. If the $m - 3$ remaining independent generators are all of order 2, at least two of them are non-commutative and generate the non-cyclic group of order 6. As an operator of order 3 contained therein could be used as an independent generator, this reduces to the preceding case and hence requires no further consideration here.

If the subgroup generated by the independent generator of order 4 is non-invariant, it is transformed into another such subgroup by an independent generator of order 3 and these two generators generate the dicyclic group of order 12. The remaining $m - 4$ independent generators can be so selected that they are of order 2 and relatively commutative since it may be assumed that no invariant cyclic subgroup of order 12 is in G. They transform this dicyclic group of order 12 into itself. As the group of isomorphisms of this dicyclic group is of order 12 and the group of its inner isomorphisms is of order 6 it results that there are two and only two such groups of order 12.2^{m-4} for every value of $m > 4$. One of these two groups is the direct product of the dicyclic group of order 12 and the abelian group of order 2^{m-4} and of type 1^{m-4} while the other is the direct product of the group of order 24 obtained by extending the dicyclic group of order 12 by an operator of order 2 which transforms each of its operators of order 4 into its inverse and the abelian group of order 2^{m-5} and of type 1^{m-5}.

When the $m - 2$ independent generators of G involve necessarily more than one whose order is 4 every two of these operators generate the quaternion group and hence all of them have a common square. Hence we may assume that G has only two independent generators of order 4 and involves invariantly the quaternion group generated by them. If all of the remaining generators are of order 2, two of them are non-commutative and generate the non-cyclic group of order 6. Hence it may be assumed that one of the independent generators of G is of order 3 and that G is the direct product of the Hamiltonian group whose order is a power of 2 and the group of order 3. It may be noted that every set of independent generators of this Hamiltonian group involves at least two of order 4 but that these independent generators can be so selected that an arbitrary larger number of them have this order. If the order of the group is 2^α the number

of these generators is necessarily $\alpha - 1$. It remains to consider the possible groups when none of the $m - 2$ independent generators has an order which exceeds 3.

It was noted above that the number of independent generators of order 3 in a set of $m - 2$ such generators of G could not exceed three and when there are three such generators in the set they generate the abelian group of order 27 and of type 1^3. If the order of G exceeds 27, each of the remaining operators of the set is of order 2 and every two of these operators are commutative. They therefore generate the abelian group of order 2^{m-5} and of type 1^{m-5}. The given subgroup of order 27 is invariant, and when the Sylow subgroup of order 2^{m-5} is also invariant, G is the direct product of these two Sylow subgroups. There is one and only one such group of order 27.2^{m-5} for every value of $m > 4$. The order of the group of isomorphisms of the given invariant subgroup of order 27 is 11232 and its quotient group with respect to its invariant subgroup of order 2 is the simple group of order 5616. We are interested here only in a Sylow whose order is a power of 2 of this group of isomorphisms.[2]

The order of this Sylow subgroup is 32 and its operators of order 2 cannot generate a group whose order exceeds 8, since the given simple group contains operators of order 8. It is obvious that these operators generate the abelian group of order 8 and of type 1^3 since the operators of three generating subgroups of the given group of order 27 can be transformed into themselves or into their inverses independently of each other. Hence there results the following theorem: *If* $m > 7$ *there are* 8 *and only* 8 *groups of order* 27.2^{m-5} *which have* $m - 2$ *independent generators and whose orders are less than* 2^m. *When* $m = 7$ *there are* 7 *such groups and when* $m = 6$ *the number is* 4.

When two and only two of the $m - 2$ independent generators of G are of order 3 and they cannot be so selected that three of them are of this order then the remaining $m - 4$ of these generators are of order 2 and are relatively commutative. Suppose first that these two independent generators are contained in an invariant subgroup of order 9. As the group of isomorphisms of this subgroup is of order 48 and the quotient group with respect to its invariant subgroup of order 2 is the symmetric group of order 24 there results the following theorem: *When* $m > 5$ *there are* 4 *and only* 4 *groups of order* 9.2^{m-4} *which contain an invariant subgroup of order* 9, *have separately* $m - 2$ *independent generators and whose orders are less than* 2^m; *when* $m = 5$ *there are* 3 *such groups*. When the two independent generators are commutative they must appear in an invariant subgroup of order 9 and when they are non-commutative they must appear an invariant tetrahedral group.

In the latter case the $m - 4$ remaining independent generators appear in an abelian subgroup of order 2^{m-4} and of type 1^{m-4}. This abelian group

is either commutative with all the operators of the given tetrahedral group or transforms it according to an outer isomorphism. In the former case G is the direct product of this tetrahedral group and the abelian group of order 2^{m-4} and of type 1^{m-4}. In the latter case it is the direct product of the symmetric group of order 24 and the abelian group of order 2^{m-5} and of type 1^{m-5}. As there could not be only one operator of order 3 in the set of $m - 2$ independent generators unless these generators could be so selected that at least two of them would be of this order, the consideration of all the possible cases is thus completed.

[1] G. A. Miller, these PROCEEDINGS, **22**, 112 (1936).
[2] G. A. Miller, *Collected Works*, **1**, 205 (1935).

ON PRODUCTS IN A COMPLEX[1]

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1. *Introduction.*—In recent years the existence of certain products in a complex K has been much studied, combining a p-chain and a q-chain to form a $(p + q)$-chain. For such products to have topological significance, certain conditions should be satisfied; it seems that (P_1) and (P_2) below (§5) are the most natural. It turns out (Theorems 2 and 4) that such products always exist, whether K is simplicial or not, and that if K is connected,[2] each such product, when taken in the cohomology groups, is a multiple of a uniquely defined product. The usefulness of the present definition of the products is shown, for instance, by the ease with which they may be defined in product complexes (§8), and by their use in the proof of Theorem 6. As an application, a mapping theorem of Hopf is given (§9).

The products of §3 were discovered independently by E. Čech[3] and the author; the formulas (11) were also discovered by the author at the end of 1935, and were applied in the classification of the maps of a 3-complex into a 2-sphere.[4] The methods of this paper are based largely on some ideas of Čech (loc. cit.); the results will be published in full elsewhere.

We shall use ∂ and δ for the boundary and coboundary operators, and shall write $A \sim B$ or $A \backsim B$ if A is homologous to B or cohomologous to B. We shall use I for the 0-chain Σx_i = sum of all the vertices. We assume, about the structure of K, that the homology groups of each closed cell vanish.

Remark.—(Added in proof.) A very general type of complex may be used. It is simplest to construct first the $\frown$ product, then the $\smile$ product.

2. *Dual Homomorphisms.*—Let L^r be the group of r-chains of K, generated by the r-cells σ_i^r of K. Given any two such groups L^r and L'^s, there is a (1–1) correspondence between homomorphisms ϕ and ϕ' of L^r into L'^s and of L'^s into L^r; if ϕ is defined by $\phi(\sigma_i^r) = \Sigma\phi_{ij}\sigma_j^s$, then ϕ' is defined by $\phi'(\sigma_i^s) = \Sigma\phi_{ji}\sigma_j^r$. We call these *dual homomorphisms*, and write $\phi' = D(\phi)$. Clearly

$$D(\phi\Psi) = D(\Psi)D(\phi). \tag{1}$$

Forming linear combinations of homomorphisms, we have

$$D(a\phi + b\Psi) = aD(\phi) + bD(\Psi). \tag{2}$$

Note that ∂ and δ are dual. If $\phi' = D(\phi)$, then

$$\delta\phi' A'' = \phi'\delta A'' \text{ if and only if } \partial\phi A^r = \phi\partial A^r, \tag{3}$$

in each case for all r-chains A'' and A^r, by (1). These relations hold if ϕ is defined by a simplicial map. If $(\Sigma\alpha_i\sigma_i^r)\cdot(\Sigma\beta_i\sigma_i^r) = \Sigma\alpha_i\beta_i$, then clearly

$$A\cdot\phi'\beta' = \phi A\cdot B' \text{ if and only if } \phi' = D(\phi). \tag{4}$$

3. *The Products in Simplicial Complexes.*—Let K be simplicial, and let its vertices be ordered in a fixed manner. Then any r-chain may be written uniquely in the form

$$\Sigma a_{\alpha_0\ldots\alpha_r}x_{\alpha_0}\ldots x_{\alpha_r},\ \alpha_0 < \ldots < \alpha_r.$$

We define two products by

$$(\Sigma a_{\alpha_0\ldots\alpha_r}x_{\alpha_0}\ldots x_{\alpha_r})\smile_0(\Sigma b_{\alpha_0\ldots\alpha_s}x_{\alpha_0}\ldots x_{\alpha_s}) = \Sigma a_{\alpha_0\ldots\alpha_r}b_{\alpha_r\ldots\alpha_{r+s}}x_{\alpha_0}\ldots x_{\alpha_{r+s}}, \tag{5}$$

$$(\Sigma a_{\alpha_0\ldots\alpha_r}x_{\alpha_0}\ldots x_{\alpha_r}) \frown_0 (\Sigma b_{\alpha_0\ldots\alpha_s}x_{\alpha_0}\ldots x_{\alpha_s}) = \Sigma a_{\alpha_{s-r}}\ldots a_{\alpha_s}b_{\alpha_0\ldots\alpha_s}x_{\alpha_0}\ldots x_{\alpha_{s-r}}. \tag{6}$$

These are related as follows. If

$$\phi_j^s A^r = A^r \smile_0 \sigma_j^s,\ \phi'^s B^{r+s} = \sigma_j^s \frown_0 B^{r+s}, \text{ then } \phi'^s_j = D(\phi_j^s). \tag{7}$$

The products satisfy the dual relations (compare Alexander, loc. cit.)

$$\delta(A^r \smile_0 B^s) = \delta A^r \smile_0 B^s + (-1)^r A^r \smile_0 \delta B^s, \tag{8}$$

$$\partial(A^r \frown_0 B^s) = (-1)^{s-r}\delta A^r \frown_0 B^s + A^r \frown_0 \partial B^s. \tag{9}$$

Hence corresponding products are defined in the cohomology and homology groups. Clearly

$$(A \smile_0 B) \smile_0 C = A \smile_0 (B \smile_0 C), \quad (A \smile_0 B) \frown_0 C = A \frown_0 (B \frown_0 C). \tag{10}$$

For the commutative law, see §5.

4. *Products and Simplicial Maps.*—Let K be mapped simplicially into K', and let ϕ and ϕ' be the corresponding homomorphisms. Then if the vertices of K are ordered so that the ordering is not altered when ϕ is applied, it is easy to see that the dual relations

$$\phi'(A' \smile_0 B') = \phi' A' \smile_0 \phi' B', \quad \phi(\phi' A' \frown_0 B) = A' \frown_0 \phi B \tag{11}$$

hold. These generalize a theorem of Hopf.[5]

5. *The general $\smile$ products.*—We consider all bilinear operations satisfying (St = star)

(P_1) $\sigma_i^r \smile \sigma_j^s$ is an $(r+s)$-chain in $St(\sigma_i^r) \cdot St(\sigma_j^s)$,

(P_2) $\delta(A^r \smile B^s) = \delta A^r \smile B^s + (-1)^r A^r \smile \delta B^s$.

THEOREM 1. *If K is connected, then for any such product there is an integer $\gamma(\smile)$ such that*

$$I \smile A^r = A^r \smile I = \gamma(\smile) A^r. \tag{12}$$

The important case is $\gamma = 1$; this is so for $\smile_0$(§3).

THEOREM 2. *For any integer γ, such a product may be constructed in any complex.* Compare (15).

THEOREM 3. *If $\gamma(\smile) = 0$, then a bilinear product $\vee$ may be constructed satisfying*

(Q_1) $A^\circ \vee B^s = 0$,

(Q_2) If $r > 0$, $\sigma_i^r \vee \sigma_j^s$ is an $(r+s-1)$-chain in $St(\sigma_i^r) \cdot St(\sigma_j^s)$,

(Q_3) $\sigma_i^r \smile \sigma_j^s = \delta(\sigma_i^r \vee \sigma_j^s) + \delta\sigma_i^r \vee \sigma_j^s + (-1)^r \sigma_i^r \vee \delta\sigma_j^s$.

The method of construction of both $\smile$ and $\vee$ is the same. Having constructed $A^r \smile B^s$ (all s) for $p \leqq r$, we construct $A^{r+1} \smile B^s$ in each closed $(r+s+1)$-cell separately; this defines it in the whole complex.

We may form linear combinations of products:

$$A(a \smile_1 + b \smile_2) B = a(A \smile_1 B) + b(A \smile_2 B). \tag{13}$$

Clearly $\gamma(a \smile_1 + b \smile_2) = a\gamma(\smile_1) + b\gamma(\smile_2)$.

THEOREM 4. *Suppose $\gamma(\smile_1) = 1$. Then for any $\smile$, the two products $\smile$ and $\gamma(\smile)\smile_1$ give the same product in the cohomology groups.*

To prove this, set $\smile' = \smile - \gamma(\smile)\smile_1$. Then $\gamma(\smile') = 0$, and we may construct $\vee'$ by Theorem 3. If A and B are cocycles, (Q_3) shows that $A \smile' B \backsim 0$; hence $A \smile B \backsim \gamma(\smile) A \smile_1 B$.

The commutative law $A^r \smile B^s \backsim (-1)^{rs} B^s \smile A^r$ (for cocycles) is proved by setting $A^r \smile' B^s = (-1)^{rs} B^s \smile A^r$ and showing that (P_1) and (P_2) hold; clearly $\gamma(\smile') = \gamma(\smile)$. The associative law, for simplicial complexes, follows from (10); in the general case, we must use an invariance theorem (§7).

6. *The General $\frown$ Products.*—We now assume

(R_1) $\sigma_i^r \frown \sigma_j^s$ $(r \leqq s)$ is an $(s - r)$-chain in $\overline{St(\sigma_i^r) \cdot \sigma_j^s}$.

(R_2) $\partial(A^r \frown B^s) = (-1)^{s-r} \delta A^r \frown B^s + A^r \frown \partial B^s$.

There is a (1–1) correspondence between these products and the former ones. If K is connected, there is an integer γ such that

$$I \frown A^r = \gamma A^r; \tag{14}$$

corresponding $\smile$ and $\frown$ products have the same γ. The theorem corresponding to Theorem 3 is proved by taking an operation dual to $\vee$ (compare (7)). Other properties of $\frown$ are similar to those of $\smile$.

Hereafter we assume $\gamma(\smile) = \gamma(\frown) = 1$.

7. *Invariance Theorems.*—Given any complex K, let K' be a simplicial subdivision. Let $\bar{\sigma}_k(\tau)$ be the cell of lowest dimension of K containing the cell τ of K'. Each chain A of K corresponds to its subdivision SdA in K'. Recall that $\partial SdA = Sd\partial A$; hence, taking duals, $Sd'\delta A' = \delta Sd'A'$. We state

THEOREM 5. *There are homomorphisms ϕ of L'^r into L^r and ψ of L'^r into L^{r+1} such that*

(a) $\phi(\tau^r)$ and $\psi(\tau^r)$ are in $\bar{\sigma}_k(\tau^r)$,
(b) $\partial\phi A'^r = \phi\partial A'^r$,
(c) $\phi SdA^r = A^r$,
(d) $Sd\phi A'^r = A'^r + \partial\psi A'^r + \psi\partial A'^r$.

If K is simplicial, we may let ϕ be any pseudo-identical map; otherwise, ϕ is easily constructed step by step. When (a) and (b) hold, so does (c). ψ is now easily constructed so as to satisfy (d).

THEOREM 6. *The homomorphism ϕ of Theorem 5 and its dual ϕ' induce isomorphisms between the homology and cohomology groups of* K *and* K'; *they preserve the definitions of $\smile$ and $\frown$.*

The first part is an immediate consequence of the last theorem. To prove that $\smile$ is preserved, choose a $\smile$ product in K' (for instance that of §3), and define a new $\smile$ product in K by

$$A \smile' B = Sd'(\phi' A \smile \phi' B). \tag{15}$$

It is easily verified that (P_1) and (P_2) hold, and that $\gamma(\smile') = 1$, as required. Using the dual of (d) above shows that, for cocycles A and B,

$$\phi'(A \smile' B) = \phi' Sd'(\phi' A \smile \phi' B) \backsim \phi' A \smile \phi' B, \tag{16}$$

as required. The relation for the $\frown$ product follows from this by considering duals.

The theorem on *topological invariance* now follows as usual; or we could define the groups and products in an abstract space, and show that they reduce to the above for complexes.

8. *Product Complexes.*—Given $\smile$ and $\frown$ products in K_1 and K_2, we define them in $K_1 \times K_2$ by

$$(\sigma^p \times \tau^r) \smile (\sigma^q \times \tau^s) = (-1)^{qr}(\sigma^p \smile \sigma^q) \times (\tau^r \smile \tau^s), \quad (17)$$

$$(\sigma^p \times \tau^r) \frown (\sigma^q \times \tau^s) = (-1)^{p(s-r)}(\sigma^p \frown \sigma^q) \times (\tau^r \frown \tau^s). \quad (18)$$

It is easily seen that these satisfy the required conditions.

Consider Euclidean n-space, subdivided by means of planes with integral coördinates. The cells are

$$(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n),\ \alpha_i \text{ integral},\ \beta_i = 0 \text{ or } 1; \quad (19)$$

it denotes the cell with $\alpha_i \leqq x_i \leqq \alpha_i + \beta_i$ $(i = 1, \ldots, n)$, and is of dimension $\Sigma\beta_i$. Other symbols denote 0. Set

$$P(\beta;\ \gamma) = (-1)^a,\ a = \sum_{i<j} \gamma_i\beta_j. \quad (20)$$

Then, using (17) and (18), applied to the products of §3 in 1-space,

$$(\alpha_1, \beta_1; \ldots) \smile (\alpha_1 + \beta_1, \gamma_1; \ldots) = P(\beta;\ \gamma)(\alpha_1, \beta_1 + \gamma_1; \ldots), \quad (21)$$

$$(\alpha_1 + \gamma_1 - \beta_1, \beta_1; \ldots) \frown (\alpha_1, \gamma_1; \ldots) = P(\gamma - \beta; \beta)(\alpha_1, \gamma_1 - \beta_1; \ldots). \quad (22)$$

It is not hard to verify the required properties directly.

9. *On the Maps of* $S^n \times S^n$ *into* S^n_o. Take a simplicial subdivision of the n-sphere S^n, let Z^n be the fundamental cycle, and let σ^n be an n-simplex whose first vertex is x. Set $Z^{2n} = Z^n \times Z^n$, $\sigma^{2n} = \sigma^n \times \sigma^n$. Define the two cycles and two cocycles

$$A^n = Z^n \times x,\ B^n = x \times Z^n,\ C^n = \sigma^n \times I,\ D^n = I \times \sigma^n. \quad (23)$$

We find at once, by (17),

$$C^n \smile C^n = D^n \smile D^n = 0,\ C^n \smile D^n = \sigma^{2n},\ D^n \smile C^n = (-1)^n\sigma^{2n}, \quad (24)$$

$$A^n \cdot C^n = B^n \cdot D^n = 1,\ A^n \cdot D^n = B^n \cdot C^n = 0. \quad (25)$$

A^n and B^n [C^n and D^n] generate the n-dimensional homology [cohomology] group of $S^n \times S^n$.[6] Now let f be a simplicial map of $S^n \times S^n$ into S^n_o, with dual f'. For an n-cell σ^n_o of S^n_o, $\delta\sigma^n_o = 0$; hence if $E^n = f'\sigma^n_o$, $\delta E^n = f'\delta\sigma^n_o = 0$, and $E^n \sim \alpha C^n + \beta D^n$. By (11) and (24),

$$0 = f'(0) = f'(\sigma^n_o \smile \sigma^n_o) = E \smile E \backsim [1 + (-1)^n]\alpha\beta\sigma^{2n}.$$

As σ^{2n} is not $\backsim$ 0, it follows that *if* n *is even, then* $\alpha\beta = 0$. By (4) and (25), f maps A^n and B^n into S_o^n with the degrees α and β; this proves a theorem of Hopf.[7]

Appendix.—10. *Products in Terms of Other Operations.*—Define

$$(\sum \alpha_i \sigma_i^r) \circ (\sum \beta_i \sigma_i^r) = \sum \alpha_i \beta_i \sigma_i^r. \tag{26}$$

In a simplicial complex, with ordered vertices, define homomorphisms of L^s into L^r ($s \geqq r$) by

$$u^r(x_{i_0} \ldots x_{i_s}) = x_{i_0} \ldots x_{i_r}, \quad v^r(x_{i_0} \ldots x_{i_s}) = x_{i_{s-r}} \ldots x_{i_s}. \tag{27}$$

Let U^s and V^s be the dual homomorphisms. Then clearly (see §3)

$$A^r \smile_0 B^s = U^{r+s} A^r \circ V^{r+s} B^s, \quad A^r \frown_0 B^s = u^{s-r}(V^s A^r \circ B^s). \tag{28}$$

If $I^r = \sum \sigma_i^r$, then $\delta I^{2r} = 0$, $\delta I^{2r+1} = I^{2r+2}$; also

$$A^r \circ I^r = I^r \circ A^r = A^r, \quad U^s I^r = V^s I^r = I^s \quad (s \geqq r). \tag{29}$$

Note that $A^r \circ K^r$ is the part of A^r in the subcomplex K^r, in an obvious sense.

11. *Tensor products of groups.*—The definition of "group pairs" may also be split into two parts, as follows. Given the (abelian) groups G and H, the *tensor product* GH is the group with generators gh (defined abstractly), and, for relations, all those obtainable with the two distributive laws.[8] Then the possible definitions of group pairs G and H with respect to the group Z are obtained merely by choosing the possible homomorphisms of GH into Z.

If we wish to study products of L^r and L^s into L^{r+s}, using coefficient groups G in L^r and H in L^s, then we most naturally use the coefficient group GH in L^{r+s}; the theory with any other coefficient group in L^{r+s} follows from this.

The reason for the name "tensor product" is the following. A contravariant vector (at a point) is a point of a vector space V. If H is the group of homomorphisms of V into the real numbers, then a covariant vector is an element of H. Now the tensors $A^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ are just the elements of the tensor product $V \ldots VH \ldots H$, taking V p times and H q times.

[1] Presented to the American Mathematical Society, March 27, 1937.

[2] If K is not connected, we may consider products in each component of K separately.

[3] E. Čech, *Ann. Math.*, **37** (1936). See also J. W. Alexander, *Ann. Math.*, **37** (1936) for a study of one of these products.

[4] See *Bull. Am. Math. Soc.*, **42**, 338 (1936). The results were communicated to L. Zippin, and later (April, 1936) lectures were delivered on the subject at Harvard and Princeton. See also H. Freudenthal, *Proc. Kon. Ak. Wet., Amsterdam*, **40**, 54–60 (1937).

[5] H. Hopf, *Jour. Math.*, **163** (1930), Satz I. Compare Freudenthal, loc. cit.

[6] The latter statement follows from the former, using the relations $C^n \frown Z^{2n} = (-1)^n B^n$, $D^n \frown Z^{2n} = A^n$ (compare Čech, loc. cit.).

[7] H. Hopf, *Fund. Math.*, **25**, part of Satz V (1935).

[8] Also $(-g)\ (-h) = gh$.

LINEAR ORDERS[1]

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The considerations of this paper are based on the introduction of composite orders in relation to given ones. The reference throughout is to linear order. If α and β are any two given order properties, for example, *dense* and *scattered* (i.e., without any dense suborder), we say that an order A is of property $\alpha\beta$, if A may be formed by replacing each element of an α-order by a β-order; in other words, A is the sum of β-orders over an α-order. An order property α is said to be *iterative* if $\alpha\alpha$ implies α. Suppose α is a given order property, iterative or not. Let α^0 be the property of consisting of a single element; this α^0, of course, is the same for all order properties. By α^1 we understand α itself. Suppose the order property α^μ is defined for all ordinals μ less than λ. An order A is said to have property α^λ if it is the sum of α^μ-orders over an α-order, where $\mu < \lambda$ and μ is permissibly variable. We show that the order property α^λ, for λ sufficiently large, is iterative—i.e., there exists a first ordinal λ such that α^λ is iterative. Moreover, we show that α^λ is *minimal* in the sense that every iterative property implied by α is implied by α^λ. We call α^λ the *minimal iterative property implied by* α. In the case where α is the property of being either a normal order or reverse normal order α^λ is the property *scattered*.

We now define the order property α^* associated with an order property α. Let α' be the property of being a segment[2] of an α-order, and β the property of being either a normal order or a reverse normal order. We denote by α^* the minimal iterative property implied by $\beta\alpha'$. It is readily shown that α^* is descending (every segment of an α^*-order is an α^*-order), and that $\beta\alpha^*$ implies α^*. We call α^* *transitive*, and in general, if γ is a descending and iterative property such that $\beta\gamma$ implies γ, we call γ transitive. Furthermore, α^* is the minimal transitive property implied by α in the sense that every transitive property implied by α is implied by α^*. We now state the principal result of the paper, which gives a unique decomposition of every order with respect to every order property. *If* α

is a given order property and A *a given order, then* A *either has property* α^* *or is the sum of* α^**-orders over an order no proper segment*[3] *of which has property* α^*, *where* α^* *is the minimal transitive property implied by* α. If, in particular, α is the property of being of cardinal less than $\aleph_\lambda$, with $\aleph_\lambda$ regular,[4] α^* is the order property $\aleph_\lambda$*-scattered*, definable as follows: An order is said to be $\aleph_\lambda$*-dense* if it is of cardinal $\aleph_\lambda$ and every proper segment of it is of cardinal $\aleph_\lambda$. An order containing no $\aleph_\lambda$-dense suborder is said to be $\aleph_\lambda$-scattered. We obtain the decomposition: *Every order is either* $\aleph_\lambda$*-scattered or the sum of* $\aleph_\lambda$*-scattered orders over an* $\aleph_\lambda$*-dense order.* If, in particular, α is the property of being of cardinal less than $\aleph_0$, i.e., finite, α^* is the property *scattered* and we have the result: *Every order is either scattered or the sum of scattered orders over a dense order.*[5] Similarly, if α is the property of not containing the normal order ω_λ, with ω_λ regular, we obtain the notions ω_λ-scattered and ω_λ-dense, and a corresponding decomposition theorem. Other particular transitive properties and their associated decompositions are readily obtained.

The above considerations suggest the notions α*-dense* and α*-scattered* defined in the following manner: An order A is said to be α-dense if A is dense and every proper segment of A contains an α-order as suborder. An order not containing any α-dense suborder is said to be α-scattered. It is shown that the property α-scattered is transitive for all α. We have therefore a second decomposition of every order with respect to an order property α. The latter decomposition is the "reverse" of the former in the sense indicated by the following theorem: *If* α *is a given order property and* $\bar{\alpha}$ *the property of containing no suborder of property* α, *then* $\bar{\alpha}^*$ *is the property* α*-scattered.*

We show that every $\aleph_\lambda$-scattered order, with $\aleph_\lambda$ regular, contains either ω_λ or its reverse $\omega_\lambda{}^*$. It follows that every order of regular cardinal $\aleph_\lambda$ either contains an $\aleph_\lambda$-dense order or contains the normal order ω_λ or the reverse normal order $\omega_\lambda{}^*$. We readily obtain other theorems on general orders by means of the above decompositions and constructions.

The paper in full will be offered to the *Transactions of the American Mathematical Society.*

[1] I wish to express my thanks and appreciation to Professor H. Blumberg for his generous help in the preparation of this paper.

[2] A segment of an order is a subset S of the order such that if a and b are elements of S, every element of the order following a and preceding b is in S.

[3] A segment of an order is said to be *proper* if it contains more than one element.

[4] The cardinal $\aleph_\lambda$ and the normal order ω_λ initiating the cardinal $\aleph_\lambda$ are said to be *regular* if every suborder of ω_λ cofinal with ω_λ is of type ω_λ.

[5] Cf. Hausdorff, *Grundzüge der Mengenlehre*, 95–97 (1914).

HOPF'S THEOREM FOR NON-COMPACT SPACES

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A known theorem of Hopf's[1] for complexes, as extended to compact metric spaces by Freudenthal,[2] is to the effect that the homotopy classes of mappings of a compact metric n-space R on the n-sphere H_n are in 1–1 correspondence with the n-dimensional dual homology group of R. If the mapping class Φ corresponds to the dual homology class Γ of R, then any $f\epsilon\Phi$ maps the basic dual n-cycle of H_n into Γ. Bruschlinsky[3] proved this theorem for mappings on H_1 without restricting the dimension of R.

Making use of (a) uniform homotopy, (b) infinite dual chains, we have succeeded in extending these results in various ways. More important, however, than this extension is the following incidental result.

THEOREM 1. *The 1-dimensional Čech dual homology group of the straight line is isomorphic with the additive group of continuous real functions on the straight line reduced mod the group of bounded continuous functions.*

This is a consequence of Theorem 2 below, as can be shown by using a lemma of Eilenberg's.[4] It follows that *the Čech Betti number of the straight line is* $\aleph$. This is the first new intuitive result on the nature of the Čech homology groups of non-compact spaces.

Let I be the closed line segment $[0, 1]$. Two continuous mappings f and g of a topological space R on a metric space S are called homotopic if there exists a continuous mapping h of $R \times I$ on S such that $h(x, 0) = f(x)$, and $h(x, 1) = g(x)$ for x in R. They are called uniformly homotopic if h can be chosen with the further property that for $\epsilon > 0$ there is a δ such that if $|y_1 - y_2| < \delta$, the distance $\rho(h(x, y_1), h(x, y_2)) < \epsilon$ for x in R. If R is compact (not necessarily metric), two continuous mappings of R on a metric space are homotopic if and only if they are uniformly homotopic. The set of all mappings [uniformly] homotopic to a given one forms a [uniform] homotopy class.

THEOREM 2. *The theorems of Hopf and Bruschlinsky are true for normal spaces if we use uniform homotopy and homology in the sense of Čech.*

If R is also compact (not necessarily metric), "uniform homotopy" is equivalent to "homotopy" and this theorem becomes a direct generalization of the theorems of Hopf and Bruschlinsky.

To obtain a corresponding theorem on ordinary homotopy we introduce a new kind of dual homology group. The coverings of the fundamental family are at most countably infinite coverings by open sets such that each point of R is contained in only a finite number of the sets. The nerve is here an infinite (possibly finite) complex. An n-chain of a cover-

ing $\mathfrak{U}$ is a sum $\sum_i r^i E_i^n$, the r^i being integers and the summation being over all the n-simplexes E_i^n of $\mathfrak{U}$. The dual homology groups of R based on these infinite chains are now defined as usual.

THEOREM 3. *The theorems of Hopf and Bruschlinsky are true for metric separable spaces if we use ordinary homotopy and homology groups based on infinite chains.*

[1] Hopf, H., *Comm. Math. Helv.*, **5**, 39–54 (1933).
[2] Freudenthal, H., *Comp. Math.*, **2**, 134–176 (1935).
[3] Bruschlinsky, N., *Math. Ann.*, **109**, 525–537 (1934).
[4] Eilenberg, S., *Fund. Math.*, **26**, 68 (1936).